York University

Faculty of Arts, Faculty of Science

Math 1090

Midterm Test 1

SOLUTIONS

Instructions:

1. There are 5 questions on 5 pages.
2. Answer all questions.
3. Your work must justify the answer you give.

Question	Points	Marks
1	6	
2	6	
3	3	
4	8	
5	7	
Total	30	

1. (6 points) Prove in the style of the text, (3.43)(b),

$$\vdash p \ \vee \ (p \wedge q) \ \equiv \ p \ .$$

You may only use lower numbered theorems in your proof.
Answer:

$$
\begin{array}{ll}
 & p \ \vee \ (p \wedge q) \\
= & \langle \ (3.35) \ \rangle \\
 & p \ \vee \ (p \equiv q \equiv p \vee q) \\
= & \langle \ (3.27) \text{ twice } \rangle \\
 & p \vee p \ \equiv \ p \vee q \ \equiv \ p \vee (p \vee q) \\
= & \langle \ (3.25) \ \rangle \\
 & p \vee p \ \equiv \ p \vee q \ \equiv \ (p \vee p) \vee q \\
= & \langle \ (3.26) \text{ twice } \rangle \\
 & p \ \equiv \ p \vee q \ \equiv \ p \vee q \\
= & \langle \ (3.3) \ \rangle \\
 & p \ \equiv \ true \\
= & \langle \ (3.2) \ \rangle \\
 & true \ \equiv \ p \\
= & \langle \ (3.3) \ \rangle \\
 & p \ .
\end{array}
$$

2. (6 points) Prove in the style of the text,

$$\vdash p \wedge q \;\Rightarrow\; p \;\wedge\; (p \Rightarrow q)\ .$$

You may only use theorems numbered below (3.66) in your proof.
Answer:

$$
\begin{array}{ll}
 & p \wedge q \;\Rightarrow\; p \;\wedge\; (p \Rightarrow q) \\
= & \langle\ (3.59)\ \rangle \\
 & \neg(p \wedge q) \;\vee\; (p \wedge (\neg p \vee q)) \\
= & \langle\ (3.47)(\text{a})\ \rangle \\
 & (\neg p \vee \neg q) \;\vee\; (p \wedge (\neg p \vee q)) \\
= & \langle\ (3.45),\ (3.25)\ \text{implicitly}\ \rangle \\
 & (\neg p \vee \neg q \vee p) \;\wedge\; (\neg p \vee \neg q \vee \neg p \vee q) \\
= & \langle\ (3.24),\ (3.25)\ \text{implicitly}\ \rangle \\
 & (\neg q \vee p \vee \neg p) \;\wedge\; (\neg p \vee \neg p \vee q \vee \neg q) \\
= & \langle\ (3.26),\ (3.28),\ \text{Metatheorem (3.7)}\ \rangle \\
 & (\neg q \vee \mathit{true}) \;\wedge\; (\neg p \vee \mathit{true}) \\
= & \langle\ (3.29)\ \text{twice}\ \rangle \\
 & \mathit{true} \wedge \mathit{true} \\
= & \langle\ (3.39)\ \rangle \\
 & \mathit{true}\ .
\end{array}
$$

3. (3 points) Fill in complete reasons (in the style of the model proof) for the proof step

$$p \wedge (q \Rightarrow p) \;\equiv\; (q \Rightarrow p)$$

$$= \quad \Big\langle \qquad \qquad \qquad \qquad \qquad \qquad \Big\rangle$$

$$p \wedge (q \wedge p \;\equiv\; q) \;\equiv\; (q \Rightarrow p)\ .$$

Answer:
By (3.60), $\vdash\ (q \Rightarrow p) \;\equiv\; (q \wedge p \equiv q)$.
By Leibniz with E being $p \wedge r \;\equiv\; (q \Rightarrow p)$,
$\vdash\ (p \wedge (q \Rightarrow p) \;\equiv\; (q \Rightarrow p)) \;\equiv\; (p \wedge (q \wedge p \;\equiv\; q) \;\equiv\; (q \Rightarrow p))$.

4. (8 points) One of the following boolean expressions is a theorem and the other is not a theorem. Decide which one is a theorem and give a proof in the style of the text. Decide which one is not a theorem and explain why it is not a theorem.

   (a) $(p \wedge q \Rightarrow r) \;\Rightarrow\; (p \Rightarrow r) \;\wedge\; (q \Rightarrow r)$.

   (b) $(p \wedge q \Rightarrow r) \;\Rightarrow\; (p \Rightarrow r) \;\vee\; (q \Rightarrow r)$.

Answer:
(a) is not a theorem.
$(p \wedge q \Rightarrow r) \;\Rightarrow\; (p \Rightarrow r) \;\wedge\; (q \Rightarrow r)$ is not satisfied when p is F, q is T, r is F.
By soundness, (a) cannot be a theorem as theorems interpret to be valid in all states.

(b) is a theorem. Here is a proof.

$$
\begin{array}{ll}
 & (p \wedge q \Rightarrow r) \quad \Rightarrow \quad (p \Rightarrow r) \ \vee \ (q \Rightarrow r) \\
= & \big\langle \ (3.59) \text{ three times } \big\rangle \\
 & \neg(p \wedge q) \vee r \quad \Rightarrow \quad (\neg p \vee r) \vee (\neg q \vee r) \\
= & \big\langle \ (3.47)(\text{a}), (3.25) \text{ implicitly } \big\rangle \\
 & (\neg p \vee \neg q \vee r) \quad \Rightarrow \quad (\neg p \vee \neg q \vee r \vee r) \\
= & \big\langle \ (3.26) \ \big\rangle \\
 & (\neg p \vee \neg q \vee r) \quad \Rightarrow \quad (\neg p \vee \neg q \vee r) \ .
\end{array}
$$

5. (7 points)

(a) Prove without using the derived inference rule, Modus Ponens, that,

$$\text{If} \vdash P \wedge Q \text{ then} \vdash P,$$

is an inference rule in our system.
Answer:

$$
\begin{array}{ll}
 & P \\
= & \langle \text{ (3.39), Derived Symmetry } \rangle \\
 & P \;\wedge\; true \\
= & \langle \text{ Metatheorem (3.7), Leibniz with } E \text{ being } p \wedge r \;\rangle \\
 & P \;\wedge\; (P \wedge Q) \\
= & \langle \text{ (3.7) } \rangle \\
 & (P \wedge P) \;\wedge\; Q \\
= & \langle \text{ (3.38), Leibniz with } E \text{ being } r \wedge Q \;\rangle \\
 & P \wedge Q \;.
\end{array}
$$

By Transitivity applied three times, $\vdash\ P \equiv P \wedge Q$.
But $\vdash\ P \wedge Q$ so that by Equanimity, $\vdash\ P$.

(b) Extending the proof in (a) we could prove that if $\vdash P \wedge Q$ then $\vdash P$ and $\vdash Q$.
Show by judicious choice of P and Q that, if $\vdash P \vee Q$ then $\vdash P$ or $\vdash Q$, **is not an inference rule** in our system.
Answer:
If we take P to be p and Q to be $\neg p$ we have

$$\vdash p \vee \neg p$$

but neither p nor $\neg p$ are theorems.

The end