Math 1090 A Homework 2 due October 11 at Noon
SOLUTIONS

1. (Dow) Give complete justification (in the proof style of the "model proof") using (3.10) and lower numbered results for this proof that

$$\vdash\ (\neg(p \equiv p)\ \equiv\ (q \equiv q))\ \equiv\ false\ .$$

Answer:

$$
\begin{array}{ll}
& (\neg(p \equiv p)\ \equiv\ (q \equiv q))\ \equiv\ false \\
= & \left\langle \begin{array}{l} \text{(3.3) gives } \vdash true \equiv (p \equiv p). \text{ Derived Symmetry gives } \vdash (p \equiv p) \equiv true. \\ \text{Leibniz with } E \text{ being } (\neg r \equiv (q \equiv q)) \equiv false \text{ gives} \\ \vdash ((\neg(p \equiv p)\ \equiv\ (q \equiv q))\ \equiv\ false) \equiv ((\neg true\ \equiv\ (q \equiv q))\ \equiv\ false)\ . \end{array} \right\rangle \\
& (\neg true\ \equiv\ (q \equiv q))\ \equiv\ false \\
= & \left\langle \begin{array}{l} \text{(3.2) gives } \vdash\ \neg true \equiv (q \equiv q) \equiv (q \equiv q) \equiv \neg true\ . \\ \text{Leibniz with } E \text{ being } r \equiv false \text{ gives} \\ \vdash ((\neg true\ \equiv\ (q \equiv q))\ \equiv\ false) \equiv (((q \equiv q)\ \equiv\ \neg true)\ \equiv\ false)\ . \end{array} \right\rangle \\
& ((q \equiv q)\ \equiv\ \neg true)\ \equiv\ false \\
= & \left\langle \begin{array}{l} \text{(3.1) gives} \\ \vdash (((q \equiv q)\ \equiv\ \neg true)\ \equiv\ false) \equiv ((q \equiv q)\ \equiv\ (\neg true\ \equiv\ false))\ . \end{array} \right\rangle \\
& (q \equiv q)\ \equiv\ (\neg true\ \equiv\ false) \\
= & \left\langle \begin{array}{l} \text{(3.8) gives } false \equiv \neg true\ . \text{ Derived Symmetry gives } \vdash \neg true \equiv false. \\ \text{Leibniz with } E \text{ being } (q \equiv q) \equiv (r \equiv false) \text{ gives} \\ \vdash ((q \equiv q)\ \equiv\ (\neg true\ \equiv\ false)) \equiv ((q \equiv q)\ \equiv\ (false\ \equiv\ false))\ . \end{array} \right\rangle \\
& (q \equiv q)\ \equiv\ (false\ \equiv\ false) \\
= & \left\langle \begin{array}{l} \text{(3.3) gives } \vdash\ true \equiv (q \equiv q)\ . \text{ Derived Symmetry gives } \vdash\ (q \equiv q) \equiv true\ . \\ \text{Leibniz with } E \text{ being } (q \equiv q) \equiv (r \equiv false) \text{ gives} \\ \vdash ((q \equiv q)\ \equiv\ (false\ \equiv\ false)) \equiv (true\ \equiv\ (false\ \equiv\ false))\ . \end{array} \right\rangle \\
& true\ \equiv\ (false\ \equiv\ false)\ .
\end{array}
$$

$true \equiv (false \equiv false)$ is (3.3). Transitivity applied four times and Equanimity give $\vdash (\neg(p \equiv p)\ \equiv\ (q \equiv q))\ \equiv\ false$.

2. Using
   (3.1) $\vdash ((P \equiv Q) \equiv R) \equiv (P \equiv (Q \equiv R))$
   (3.2) $\vdash (P \equiv Q) \equiv (Q \equiv P)$
   (3.1)′ $\vdash (P \equiv (Q \equiv R)) \equiv ((P \equiv Q) \equiv R)$
   exactly as stated above, carefully prove

$$\vdash ((p \equiv q) \equiv (r \equiv s)) \equiv ((p \equiv (s \equiv q)) \equiv r) .$$

   Do not remove parentheses and do not skip steps. The required justification is the theorem number only.
   Answer:

$$\begin{array}{ll} & (p \equiv q) \equiv (r \equiv s) \\ = & \langle\ (3.2)\ \rangle \\ & (p \equiv q) \equiv (s \equiv r) \\ = & \langle\ (3.1)\ \rangle \\ & p \equiv (q \equiv (s \equiv r)) \\ = & \langle\ (3.1)'\ \rangle \\ & p \equiv ((q \equiv s) \equiv r) \\ = & \langle\ (3.2)\ \rangle \\ & p \equiv ((s \equiv q) \equiv r) \\ = & \langle\ (3.1)'\ \rangle \\ & (p \equiv (s \equiv q)) \equiv r \end{array}$$

3. Using the proof style of the text, prove (3.52),

$$\vdash (p \equiv q) \equiv (p \wedge q) \vee (\neg p \wedge \neg q) .$$

   You may assume any result with lower number.
   Answer:

$$\begin{array}{ll} & (p \wedge q) \vee (\neg p \wedge \neg q) \\ = & \langle\ (3.47)(\text{b})\ \rangle \\ & (p \wedge q) \vee \neg(p \vee q) \\ = & \langle\ (3.32)\ \rangle \\ & (p \wedge q) \vee (p \vee q) \equiv p \wedge q \\ = & \langle\ (3.25)\ \rangle \\ & ((p \wedge q) \vee p) \vee q \equiv p \wedge q \\ = & \langle\ (3.24),\ (3.43)(\text{b})\ \rangle \\ & p \vee q \equiv p \wedge q \\ = & \langle\ 3.35)\ \rangle \\ & p \equiv q . \end{array}$$

4. (a) Using the proof style of the text, prove

$$\vdash\ (p \Rightarrow q)\ \wedge\ (r \Rightarrow s)\ \ \Rightarrow\ \ (p \vee r\ \Rightarrow\ q \vee s)\ .$$

Answer:

$$
\begin{array}{ll}
 & (p \Rightarrow q)\ \wedge\ (r \Rightarrow s)\ \ \Rightarrow\ \ (p \vee r\ \Rightarrow\ q \vee s) \\
= & \langle\ (3.65)\ \rangle \\
 & (p \Rightarrow q)\ \wedge\ (r \Rightarrow s)\ \wedge\ (p \vee r)\ \ \Rightarrow\ \ q \vee s \\
= & \langle\ (3.36),\ (3.37)\ \rangle \\
 & (p \vee r)\ \wedge\ (p \Rightarrow q)\ \wedge\ (r \Rightarrow s)\ \ \Rightarrow\ \ q \vee s \\
= & \langle\ (3.46)\ \rangle \\
 & (p\ \wedge\ (p \Rightarrow q)\ \wedge\ (r \Rightarrow s)) \vee (r\ \wedge\ (p \Rightarrow q)\ \wedge\ (r \Rightarrow s))\ \ \Rightarrow\ \ q \vee s \\
= & \langle\ (3.66)\ \rangle \\
 & (p\ \wedge\ q\ \wedge\ (r \Rightarrow s)) \vee (r\ \wedge\ s\ \wedge\ (p \Rightarrow q))\ \ \Rightarrow\ \ q \vee s \\
= & \langle\ (3.78)\ \rangle \\
 & (p\ \wedge\ q\ \wedge\ (r \Rightarrow s)\ \ \Rightarrow\ \ q \vee s)\ \wedge\ \ (r\ \wedge\ s\ \wedge\ (p \Rightarrow q)\ \ \Rightarrow\ \ q \vee s) \\
= & \langle\ \text{Lemma: } \vdash\ P \wedge Q\ \Rightarrow P \vee R\ \rangle \\
 & \mathit{true}\ \wedge\ \mathit{true} \\
= & \langle\ (3.38)\ \rangle \\
 & \mathit{true}
\end{array}
$$

Proof of Lemma:

$$
\begin{array}{ll}
 & P \wedge Q\ \Rightarrow P \vee R \\
= & \langle\ (3.57)\ \rangle \\
 & (P \wedge Q) \vee (P \vee R)\ \equiv\ P \vee R \\
= & \langle\ (3.25)\ \rangle \\
 & ((P \wedge Q) \vee P) \vee R\ \equiv\ P \vee R \\
= & \langle\ (3.24),\ (3.43)(\text{b})\ \rangle \\
 & P \vee R\ \equiv\ P \vee R
\end{array}
$$

(b) Is $(p \Rightarrow q)\ \wedge (r \Rightarrow s)\ \ \equiv\ \ (p \vee r\ \Rightarrow\ q \vee s)$ a theorem? Carefully justify your answer.
Answer: It is **not a theorem** as it is not a tautology. If p, r, s are in state t, q is in state f, the expression $(p \Rightarrow q)\ \wedge (r \Rightarrow s)\ \ \equiv\ \ (p \vee r\ \Rightarrow\ q \vee s)$ is in state f.