# Math 1090 A Homework 3 due November 1 at Noon

1. (a) As (3.83) is an Axiom, it is permissible to use it in proofs of theorems with lower numbers. Using (3.83), prove each of (3.82)(b) and (3.82)(c).
**Answer:**

$$
\begin{aligned}
& (p \equiv q) \ \wedge \ (q \Rightarrow r) \quad \Rightarrow \quad (p \Rightarrow r) \\
= \quad & \langle \ (3.65) \ \rangle \\
& (p \equiv q) \quad \Rightarrow \quad ((q \Rightarrow r) \ \Rightarrow \ (p \Rightarrow r)) \ .
\end{aligned}
$$

For (3.82)(b) it suffices to prove $\vdash \ (p \equiv q) \quad \Rightarrow \quad ((q \Rightarrow r) \ \Rightarrow \ (p \Rightarrow r))$. Recall that the proof in class of the extended method of Section 4.1 did not use (3.82).

$$
\begin{aligned}
& (p \equiv q) \\
\Rightarrow \quad & \langle \ (3.83) \ \rangle \\
& (q \Rightarrow r) \ \equiv \ (p \Rightarrow r) \\
= \quad & \langle \ (3.80) \ \rangle \\
& ((q \Rightarrow r) \ \Rightarrow (p \Rightarrow r)) \ \wedge \ ((p \Rightarrow r) \ \Rightarrow (q \Rightarrow r)) \\
\Rightarrow \quad & \langle \ (3.76)(b) \ \rangle \\
& ((q \Rightarrow r) \ \Rightarrow (p \Rightarrow r))
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& (p \Rightarrow q) \ \wedge \ (q \equiv r) \quad \Rightarrow \quad (p \Rightarrow r) \\
= \quad & \langle \ (3.36), (3.65) \ \rangle \\
& (q \equiv r) \quad \Rightarrow \quad ((p \Rightarrow q) \ \Rightarrow \ (p \Rightarrow r)) \ .
\end{aligned}
$$

For (3.82)(c) it suffices to prove $\vdash \ (q \equiv r) \quad \Rightarrow \quad ((p \Rightarrow q) \ \Rightarrow \ (p \Rightarrow r))$.

$$
\begin{aligned}
& (q \equiv r) \\
\Rightarrow \quad & \langle \ (3.83) \ \rangle \\
& (p \Rightarrow q) \ \equiv \ (p \Rightarrow r) \\
= \quad & \langle \ (3.80) \ \rangle \\
& ((p \Rightarrow q) \ \Rightarrow (p \Rightarrow r)) \ \wedge \ ((p \Rightarrow r) \ \Rightarrow (p \Rightarrow q)) \\
\Rightarrow \quad & \langle \ (3.76)(b) \ \rangle \\
& ((p \Rightarrow q) \ \Rightarrow (p \Rightarrow r))
\end{aligned}
$$

(b) Prove carefully that,

$$\vdash (x = y) \;\; \Rightarrow \;\; ((y = 1) \wedge (x = 2) \;\; \Rightarrow \;\; (1 = 2)) \; .$$

Use Chapter 3 methods only.

**Answer:**

$$\begin{array}{cl} & (x = y) \;\; \Rightarrow \;\; ((y = 1) \wedge (x = 2) \;\; \Rightarrow \;\; (1 = 2)) \\ = & \langle \; (3.84)(\text{a}) \; \rangle \\ & (x = y) \;\; \Rightarrow \;\; ((x = 1) \wedge (x = 2) \;\; \Rightarrow \;\; (1 = 2)) \\ = & \langle \; (3.84)(\text{c}) \; \rangle \\ & (x = y) \;\; \Rightarrow \;\; ((x = 1) \wedge (x = 2) \;\; \Rightarrow \;\; (x = 2)) \\ = & \langle \; (3.76)(\text{b}), \text{ if } \vdash P \text{ then } \vdash P \equiv \mathit{true} \; \rangle \\ & (x = y) \;\; \Rightarrow \;\; \mathit{true} \\ = & \langle \; (3.72) \; \rangle \\ & \mathit{true} \end{array}$$

2. Using the Deduction Theorem and the method of Section 4.1 prove that,

$$\vdash (p \Rightarrow q) \;\wedge\; (r \Rightarrow s) \;\; \Rightarrow \;\; (p \wedge r \;\; \Rightarrow \;\; q \wedge s) \; .$$

**Answer:** Assume $(p \Rightarrow q) \;\wedge\; (r \Rightarrow s)$ from which one obtains temporary theorems, $p \Rightarrow q$ and $p \wedge q \equiv p$, $r \Rightarrow s$ and $r \wedge s \equiv r$.

$$\begin{array}{cl} & p \wedge r \;\; \Rightarrow \;\; q \wedge s \\ = & \langle \; \text{Temporary theorems} \; \rangle \\ & p \wedge q \wedge r \wedge s \;\; \Rightarrow \;\; q \wedge s \\ = & \langle \; (3.36), (3.37) \; \rangle \\ & (p \wedge r) \;\wedge\; (q \wedge s) \;\; \Rightarrow \;\; q \wedge s \; . \end{array}$$

3. Note that Case Analysis is the Inference Rule,

$$\frac{\vdash P \Rightarrow Q, \; \vdash \; R \Rightarrow Q}{\vdash P \vee R \Rightarrow Q} \; .$$

Using the Deduction Theorem and Case Analysis prove that,

$$\vdash (p \Rightarrow q) \;\wedge\; (r \Rightarrow s) \;\; \Rightarrow \;\; (p \vee r \;\; \Rightarrow \;\; q \vee s) \; .$$

You may use other Chapter 4 methods as you see fit in your proof.

**Answer:** Assume $(p \Rightarrow q) \;\wedge\; (r \Rightarrow s)$ from which one obtains temporary theorems, $p \Rightarrow q$ and $r \Rightarrow s$.

We need to prove $p \vee r \;\Rightarrow\; q \vee s$. By Case Analysis it suffices to prove $p \;\Rightarrow\; q \vee s$ and $r \;\Rightarrow\; q \vee s$. We have

$$\begin{array}{ll} & p \\ \Rightarrow & \langle \text{ Temporary theorem } \rangle \\ & q \\ \Rightarrow & \langle \text{ (3.76)(a) } \rangle \\ & q \vee s \end{array}$$

and

$$\begin{array}{ll} & r \\ \Rightarrow & \langle \text{ Temporary theorem } \rangle \\ & s \\ \Rightarrow & \langle \text{ (3.76)(a) } \rangle \\ & q \vee s \;. \end{array}$$

4. Determine whether each of the following as a theorem. If yes, give a proof. If no, give an interpretation for which it is false.
   Note that the variable $x$ is of type $\mathbb{N}$ and 0, 1, and 5 are constant symbols with $0 \neq 1$, etc.
   (a) $(\forall x \mid (x = 0) \wedge (x = 1) : x^2 = 5)$.
   **Answer:** This is a theorem.

$$\begin{array}{ll} & (\forall x \mid (x = 0) \wedge (x = 1) : x^2 = 5) \\ = & \langle \text{ (3.84)(a) } \rangle \\ & (\forall x \mid (x = 0) \wedge (0 = 1) : x^2 = 5) \\ = & \langle \text{ Arithmetic, i.e., } \vdash\; 0 \neq 1 \text{, if } \vdash \neg P \text{ then } \vdash P \equiv \mathit{false} \text{ } \rangle \\ & (\forall x \mid (x = 0) \wedge \mathit{false} : x^2 = 5) \\ = & \langle \text{ (3.40) } \rangle \\ & (\forall x \mid \mathit{false} : x^2 = 5) \\ = & \langle \text{ (8.13) } \rangle \\ & \mathit{true}\;. \end{array}$$

   (b) $(\exists x \mid (x = 0) \wedge (x = 1) : x^2 = 5)$.
   **Answer:** This is not a theorem. Take as Universe of Discourse $\{0, 1, 5\}$.
   When $x$ is 0, $x = 0 \;\wedge\; x = 1$ is $f$ and when $x$ is 1, $x = 0 \;\wedge\; x = 1$ is $f$ and when $x$ is 5, $x = 0 \;\wedge\; x = 1$ is $f$.
   There are no values satisfying the range condition for which $x^2 = 5$ is $t$.
   (c) $(\forall x \mid (x = 0) \vee (x = 1) : x^2 = 1)$.
   **Answer:** This is not a theorem. Take as Universe of Discourse $\{0, 1\}$.
   When $x$ is 0, $x = 0 \;\vee\; x = 1$ is $t$ but $x^2 = 1$ is $f$.

(d) $(\exists x \mid (x = 0) \vee (x = 1) : x^2 = 1)$.
**Answer:** This is a theorem.

$$
\begin{array}{cl}
 & (\exists x \mid (x = 0) \vee (x = 1) : x^2 = 1) \\
= & \langle\ (8.18)\ \rangle \\
 & (\exists x \mid x = 0 : x^2 = 1)\ \vee\ (\exists x \mid x = 1 : x^2 = 1) \\
= & \langle\ (8.14),\ x \text{ d.n.o.f. } 0,\ 1\ \rangle \\
 & 0^2 = 1 \vee 1^2 = 1 \\
= & \langle\ \text{Arithmetic, i.e., } \vdash 1^2 = 1, \text{ if } \vdash P \text{ then } \vdash P \equiv \mathit{true}\ \rangle \\
 & 0^2 = 1 \vee \mathit{true} \\
= & \langle\ (3.29)\ \rangle \\
 & \mathit{true}
\end{array}
$$

(e) $(\forall x \mid (x = y) \vee (y = 5) : (x = 5) \wedge (y = 5))$.
**Answer:** This is not a theorem.
Take as Universe of Discourse $\{0, 5\}$. Assign 0 to the free occurrences of $y$.
The range $(x = y) \vee (y = 5)$ is $t$ only when $x$ is 0 but when $x$ is 0, $(x = 5) \wedge (y = 5)$ is $f$.

(f) $(\exists x \mid (x = y) \vee (y = 5) : (x = 5) \vee (y = 5))$.
**Answer:** This is not a theorem.
Take as Universe of Discourse $\{0, 5\}$. Assign 0 to the free occurrences of $y$.
The range $(x = y) \vee (y = 5)$ is $t$ only when $x$ is 0 but when $x$ is 0, $(x = 5) \wedge (y = 5)$ is $f$.