Math 1090 A Homework 4 due November 15 at Noon

1. You are given that $\{0,1\}$ is the Universe of Discourse, P is the expression "$x = 0$", and Q is the expression "$x = 1$". Determine whether each of the following is in state t or state f.

$$\begin{aligned}(\forall x| : P \Rightarrow Q) &\Rightarrow ((\forall x| : P) \Rightarrow (\forall x| : Q)) \ .\\ ((\forall x| : P) \Rightarrow (\forall x| : Q)) &\Rightarrow (\forall x| : P \Rightarrow Q) \ .\\ (\forall x| : P \Rightarrow Q) &\equiv ((\forall x| : P) \Rightarrow (\forall x| : Q)) \ .\end{aligned}$$

Answer: Observe that
$(\forall x \,|\, P \Rightarrow Q)$ is in state f,
$(\forall x \,|\, P)$ is in state f,
$(\forall x \,|\, Q)$ is in state f.
Then
$(\forall x| : P \Rightarrow Q) \Rightarrow ((\forall x| : P) \Rightarrow (\forall x| : Q))$ is in state t,
$((\forall x| : P) \Rightarrow (\forall x| : Q)) \Rightarrow (\forall x| : P \Rightarrow Q)$ is in state f,
and $(\forall x| : P \Rightarrow Q) \equiv ((\forall x| : P) \Rightarrow (\forall x| : Q))$ is in state f.

2. Show that

$$(\exists x \,|\, : P \Rightarrow Q) \ \Rightarrow \ ((\exists x \,|\, : P) \ \Rightarrow \ (\exists x \,|\, : Q))$$

is not a theorem.
Answer: We want $(\exists x \,|\, : P \Rightarrow Q)$ be be in state t while $(\exists x \,|\, : P) \Rightarrow (\exists x \,|\, : Q)$ is in state f.
Take $\{0,1\}$ as Universe of Discourse, P to be "$x = 0$" and Q to be the Boolean constant $false$.
Then
$(\exists x \,|\, : P \Rightarrow Q)$ is in state t,
$(\exists x \,|\, : P)$ is in state t,
and $(\exists x \,|\, : Q)$ is in state f, from which
$(\exists x \,|\, : P \Rightarrow Q) \ \Rightarrow \ ((\exists x \,|\, : P) \Rightarrow (\exists x \,|\, : Q))$ is in state f.
As theorems must be in state t in all interpretations,
$(\exists x \,|\, : P \Rightarrow Q) \ \Rightarrow \ ((\exists x \,|\, : P) \Rightarrow (\exists x \,|\, : Q))$ cannot be a theorem.

3. (a) Prove that,

$$\vdash (+j \,|\, 2 \le j \le n : -(j-1)^2) = (+j \,|\, 1 \le j \le n-1 : -j^2) \ .$$

Note: Do not use (8.22). Review the proof steps for (8.22) given in the text and incorporate them in your proof.
Answer: Choose k to be fresh.

$$\begin{aligned}& (+j \,|\, 2 \le j \le n : -(j-1)^2)\\ = \ & \langle \text{ (8.14), } k \text{ d.n.o.f. in } j-1 \ \rangle\\ & (+j \,|\, 2 \le j \le n : (+k \,|\, k = j-1 : -k^2))\\ = \ & \langle \text{ (8.20), } k \text{ d.n.o.f. in } 2 \le j \le n \ \rangle\\ & (+j,k \,|\, (2 \le j \le n) \wedge (k = j-1) : -k^2)\\ = \ & \langle \text{ Arithmetic } \rangle\end{aligned}$$

$$
\begin{array}{ll}
& (+j,k \mid (2 \le j \le n) \wedge (j = k+1) : -k^2) \\
= & \langle\ (3.84)(\text{a})\ \rangle \\
& (+j,k \mid (2 \le k+1 \le n) \wedge (j = k+1) : -k^2) \\
= & \langle\ \text{Arithmetic}\ \rangle \\
& (+j,k \mid (1 \le k \le n-1) \wedge (j = k+1) : -k^2) \\
= & \langle\ \vdash (+j,k|R : P) = (+k,j|R : P)\ \rangle \\
& (+k,j \mid (1 \le k \le n-1) \wedge (j = k+1) : -k^2) \\
= & \langle\ (8.20),\ j \text{ d.n.o.f. in } 1 \le k \le n-1\ \rangle \\
& (+k \mid 1 \le k \le n-1 : (+j \mid j = k+1 : -k^2)) \\
= & \langle\ (8.14),\ j \text{ d.n.o.f. in } k+1\ \rangle \\
& (+k \mid 1 \le k \le n-1 : -k^2) \\
= & \langle\ (8.21),\ k \text{ d.n.o.f. in } 1 \le j \le n-1,\ -j^2\ \rangle \\
& (+j \mid 1 \le j \le n-1 : -j^2)
\end{array}
$$

(b) Prove $\vdash\ (+j|\, R : 0) = 0$.
Hint: $0 = (+k \mid false : 0)$.
Answer: Choose k to be fresh.

$$
\begin{array}{ll}
& (+j \mid R : 0) \\
= & \langle\ (8.13)\ \rangle \\
& (+j \mid R : (+k \mid false : 0)) \\
= & \langle\ 8.20),\ k \text{ d.n.o.f. in } R\ \rangle \\
& (+j,k \mid R \wedge false : 0) \\
= & \langle\ (3.40)\ \rangle \\
& (+j,k \mid false : 0) \\
= & \langle\ (8.20),\ k \text{ d.n.o.f. in } false\ \rangle \\
& (+j \mid false : (+k \mid\ : 0)) \\
= & \langle\ (8.13)\ \rangle \\
& 0
\end{array}
$$

(c) Rewrite using the notation of Chapter 8 of the text and prove :

$$\vdash \sum_{j=1}^{n} j^2 - (j-1)^2 \ = \ n^2 \ .$$

You may use standard arithmetic "facts" such as $a - b = a + (-b)$. You may find the results in (a) and (b) useful.
Answer: In the notation of the text we must prove

$$\vdash\ (+j \mid 1 \le j \le n : j^2 - (j-1)^2) = n^2 \ .$$

$$
\begin{aligned}
& (+j \,|\, 1 \le j \le n : j^2 - (j-1)^2) \\
= \; & \langle \text{ Arithmetic } \rangle \\
& (+j \,|\, 1 \le j \le n : j^2 + (-(j-1)^2)) \\
= \; & \langle \text{ (8.15) } \rangle \\
& (+j \,|\, 1 \le j \le n : j^2) \; + \; (+j \,|\, 1 \le j \le n : -(j-1)^2) \\
= \; & \langle \text{ (8.16) twice } \rangle \\
& (+j \,|\, 1 \le j \le n-1 : j^2) \; + \; (+j \,|\, j = n : j^2) \\
& + (+j \,|\, j = 1 : -(j-1)^2) \; + \; (+j \,|\, 2 \le j \le n : -(j-1)^2) \\
= \; & \langle \text{ (8.14) twice, } j \text{ d.n.o.f. in } n, \; j \text{ d.n.o.f. in } 1 \; \rangle \\
& (+j \,|\, 1 \le j \le n-1 : j^2) \; + \; n^2 \; + \; (-0^2) \; + \; (+j \,|\, 2 \le j \le n : -(j-1)^2) \\
= \; & \langle \text{ Arithmetic, Part (a) } \rangle \\
& n^2 \; + \; (+j \,|\, 1 \le j \le n-1 : j^2) \; + \; (+j \,|\, 1 \le j \le n-1 : -j^2) \\
= \; & \langle \text{ (8.15) } \rangle \\
& n^2 \; + \; (+j \,|\, 1 \le j \le n-1 : j^2 + (-j^2)) \\
= \; & \langle \text{ Arithmetic } \rangle \\
& n^2 \; + \; (+j \,|\, 1 \le j \le n-1 : 0) \\
= \; & \langle \text{ Part (b) } \rangle \\
& n^2 \; + \; 0 \\
= \; & \langle \text{ Arithmetic } \rangle \\
& n^2 \; .
\end{aligned}
$$

(d) Using (c) and the identity $k^2 - (k-1)^2 = 2k - 1$, prove that

$$\vdash \; (+k|1 \le k \le n : 2k - 1) = n^2 \; .$$

Answer:

$$
\begin{aligned}
& (+k|1 \le k \le n : 2k - 1) \\
= \; & \langle \; 2k - 1 = k^2 - (k-1)^2 \; \rangle \\
& (+k|1 \le k \le n : k^2 - (k-1)^2) \\
= \; & \langle \text{ Part (c) } \rangle \\
& n^2 \; .
\end{aligned}
$$