Math 1090 Homework 5 due December 3 at Noon

1. Prove
$$\vdash (\forall x \mid \, : x + z \geq 5) \; \wedge \; (y + z \geq 5 \Rightarrow y \geq 5) \quad \Rightarrow \quad y \geq 5 \; .$$

   Answer:

$$\begin{array}{ll} & (\forall x \mid \, : x + z \geq 5) \; \wedge \; (y + z \geq 5 \Rightarrow y \geq 5) \\ = & \langle \; (9.13), (3.60) \; \rangle \\ & (\forall x \mid \, : x + z \geq 5) \; \wedge \; (y + z \geq 5) \; \wedge \; (y + z \geq 5 \Rightarrow y \geq 5) \\ = & \langle \; (3.66) \; \rangle \\ & (\forall x \mid \, : x + z \geq 5) \; \wedge \; (y + z \geq 5) \; \wedge \; (y \geq 5) \\ \Rightarrow & \langle \; (3.76)(\text{b}) \; \rangle \\ & y \geq 5 \; . \end{array}$$

2. Prove
$$\vdash (\exists x \mid \, : P \Rightarrow (\forall x \mid \, : P)) \; .$$

   Answer:

$$\begin{array}{ll} & (\exists x \mid \, : P \Rightarrow (\forall x \mid \, : P)) \\ = & \langle \; (3.59) \; \rangle \\ & (\exists x \mid \, : \neg P \vee (\forall x \mid \, : P)) \\ = & \langle \; (8.15) \; \rangle \\ & (\exists x \mid \, : \neg P) \vee (\exists x \mid \, : (\forall x \mid \, : P)) \\ = & \langle \; \text{Lemma: Provided } x \text{ d.n.o.f. } Q, \, (\exists x \mid \, : Q) \equiv Q. \; \rangle \\ & \neg(\forall x \mid \, : P) \vee (\forall x \mid P) \end{array}$$

   Proof of Lemma:

$$\begin{array}{ll} & (\exists x \mid \, : Q) \\ = & \langle \; (3.39) \; \rangle \\ & (\exists x \mid \, : Q \wedge \mathit{true}) \\ = & \langle \; (9.2) \; \rangle \\ & Q \; \wedge \; (\exists x \mid \, : \mathit{true}) \\ = & \langle \; (9.28), (3.57) \; \rangle \\ & Q \; \wedge \; (\mathit{true} \vee (\exists x \mid \, : \mathit{true})) \\ = & \langle \; (3.29) \; \rangle \\ & Q \; \wedge \; \mathit{true} \\ = & \langle \; (3.39) \; \rangle \\ & Q \end{array}$$

3. Prove

$$\vdash (\forall x \mid \ : x = 1) \Rightarrow (\forall y \mid \ : y = x) \ .$$

Answer: Assume $(\forall x \mid \ : x = 1)$ from which by (9.13) and Modus Ponens, $x = 1$ is a temporary theorem.

$$\begin{array}{ll} & (\forall y \mid \ : y = x) \\ = & \langle \text{ Temporary theorem, } x = 1,\ y \text{ d.n.o.f. } (\forall x \mid \ : x = 1) \ \rangle \\ & (\forall y \mid \ : y = 1) \\ = & \langle \ (8.21),\ x \text{ d.n.o.f. } \mathit{true},\ x = 1 \ \rangle \\ & (\forall x \mid \ : x = 1) \end{array}$$

4. Determine whether each of the following is a theorem. If yes, give a proof. If no, give an interpretation for which it is in state "f".

   (a) $(P \Rightarrow Q) \ \Rightarrow \ (P \Rightarrow (\exists x| : Q))$.

   (b) $(P \Rightarrow Q) \ \Rightarrow \ ((\exists x| : P) \Rightarrow (\exists x| : Q))$.

   Answer: The expression (a) is a theorem.

$$\begin{array}{ll} & (P \Rightarrow Q) \ \Rightarrow \ (P \Rightarrow (\exists x| : Q)) \\ = & \langle \ (3.65) \ \rangle \\ & (P \Rightarrow Q) \ \wedge \ P \ \Rightarrow \ (\exists x| : Q) \\ = & \langle \ (3.66) \ \rangle \\ & P \wedge \ Q \ \Rightarrow \ (\exists x| : Q) \\ = & \langle \ (3.60) \ \rangle \\ & P \wedge \ Q \ \wedge \ (\exists x| : Q) \ \equiv \ P \wedge Q \\ = & \langle \ (9.28),\ (3.60) \ \rangle \\ & P \wedge Q \ \equiv \ P \wedge Q \end{array}$$

   The expression (b) is not a theorem.
   Take $\{0, 1\}$ to be the universe of discourse. Take Q to be false, P to be $x = 0$. Assign the value 1 to the free occurrence of x.
   We have P is f, Q is f, $P \Rightarrow Q$ is t, $(\exists x| : P)$ is t, $(\exists x| : Q)$ is f, $(\exists x| : P) \Rightarrow (\exists x| : Q)$ is f.

5. Use **Metatheorem Witness** to prove this instance of (9.27),

$$(\forall z| : yz = 1 \ \Rightarrow \ z \neq 0) \ \Rightarrow \ ((\exists x| : xy = 1) \Rightarrow (\exists x| : x \neq 0)) \ .$$

   Hint: Start by using (3.65) to obtain a form to which (9.30) applies, apply (9.30) and complete a proof.
   Answer:

$$\begin{array}{ll} & (\forall z| : yz = 1 \ \Rightarrow \ z \neq 0) \ \Rightarrow \ ((\exists x| : xy = 1) \Rightarrow (\exists x| : x \neq 0)) \\ = & \langle \ (3.65) \ \rangle \end{array}$$

$$
\begin{aligned}
& (\forall z| : yz = 1 \;\Rightarrow\; z \neq 0) \;\wedge\; (\exists x| : xy = 1) \;\Rightarrow\; (\exists x| : x \neq 0) \\
= \;& \langle\ (3.65)\ \rangle \\
& (\exists x| : xy = 1) \;\Rightarrow\; ((\forall z| : yz = 1 \;\Rightarrow\; z \neq 0) \Rightarrow (\exists x| : x \neq 0))
\end{aligned}
$$

By (9.30) it suffices to prove, $wy = 1 \;\Rightarrow\; ((\forall z| : yz = 1 \;\Rightarrow\; z \neq 0) \Rightarrow (\exists x| : x \neq 0))$.

$$
\begin{aligned}
& wy = 1 \;\Rightarrow\; ((\forall z| : yz = 1 \;\Rightarrow\; z \neq 0) \Rightarrow (\exists x| : x \neq 0)) \\
= \;& \langle\ (3.65)\ \rangle \\
& wy = 1 \;\wedge\; (\forall z| : yz = 1 \;\Rightarrow\; z \neq 0) \;\Rightarrow\; (\exists x| : x \neq 0) \\
= \;& \langle\ (9.13),\ (3.60)\ \rangle \\
& wy = 1 \;\wedge\; (\forall z| : yz = 1 \;\Rightarrow\; z \neq 0) \;\wedge\; (yw = 1 \;\Rightarrow\; w \neq 0) \;\Rightarrow\; (\exists x| : x \neq 0) \\
= \;& \langle\ \text{Arithmetic},\ (3.36)\ \rangle \\
& yw = 1 \;\wedge\; (yw = 1 \;\Rightarrow\; w \neq 0) \;\wedge\; (\forall z| : yz = 1 \;\Rightarrow\; z \neq 0) \;\Rightarrow\; (\exists x| : x \neq 0) \\
= \;& \langle\ (3.66)\ \rangle \\
& yw = 1 \;\wedge\; w \neq 0 \;\wedge\; (\forall z| : yz = 1 \;\Rightarrow\; z \neq 0) \;\Rightarrow\; (\exists x| : x \neq 0) \\
= \;& \langle\ (3.60)\ \rangle \\
& yw = 1 \;\wedge\; w \neq 0 \;\wedge\; (\exists x| : x \neq 0) \;\wedge\; (\forall z| : yz = 1 \;\Rightarrow\; z \neq 0) \\
& \quad \equiv\;\; yw = 1 \;\wedge\; w \neq 0 \;\wedge\; (\forall z| : yz = 1 \;\Rightarrow\; z \neq 0) \\
= \;& \langle\ (9.28),\ (3.60)\ \rangle \\
& yw = 1 \;\wedge\; w \neq 0 \;\wedge\; (\forall z| : yz = 1 \;\Rightarrow\; z \neq 0) \\
& \quad \equiv\;\; yw = 1 \;\wedge\; w \neq 0 \;\wedge\; (\forall z| : yz = 1 \;\Rightarrow\; z \neq 0)
\end{aligned}
$$