York University

Faculty of Arts, Faculty of Science

Math 1090

Class Test 2

SOLUTIONS

Instructions:

1. Time allowed: 50 minutes
2. There are 4 questions on 4 pages.
3. Answer all questions.
4. Your work must justify the answer you give.
5. Your reasons must indicate that you have verified any conditions concerning free occurrences of variables.

Question	Points	Marks
1	7	
2	7	
3	9	
4	7	
Total	30	

1. (7 points) Use the Deduction Theorem (method of assuming the antecedent) to prove that,

$$\vdash (p \Rightarrow q \vee r) \quad \Rightarrow \quad (p \Rightarrow q) \vee (p \Rightarrow r) \ .$$

Answer: Assume $p \Rightarrow q \vee r$ from which $p \vee q \vee r \ \equiv \ q \vee r$ is a temporary theorem.

$$\begin{aligned}
& (p \Rightarrow q) \vee (p \Rightarrow r) \\
= \quad & \langle \ (3.57) \ \rangle \\
& (p \vee q \equiv q) \vee (p \vee r \equiv r) \\
= \quad & \langle \ (3.27) \ \rangle \\
& p \vee q \vee p \vee r \ \equiv \ p \vee q \vee r \ \equiv \ q \vee p \vee r \ \equiv \ q \vee r \\
= \quad & \langle \ (3.24) \ \rangle \\
& p \vee p \vee q \vee r \ \equiv \ p \vee q \vee r \ \equiv \ p \vee q \vee r \ \equiv \ q \vee r \\
= \quad & \langle \ (3.3) \ \rangle \\
& p \vee q \vee r \ \equiv \ true \ \equiv \ q \vee r \\
= \quad & \langle \ (3.3) \ \rangle \\
& p \vee q \vee r \ \equiv \ q \vee r \ .
\end{aligned}$$

2. (7 points) Prove that,

$$\vdash z = 3 \;\Rightarrow\; (\exists y \mid z = 3 \vee y = 3 : y = z) \ .$$

Answer: Here are three different proofs. The first uses Chapter 8 results only.

$$\begin{array}{ll} & z = 3 \;\Rightarrow\; (\exists y \mid z = 3 \vee y = 3 : y = z) \\ = & \langle \text{ (8.18) as } \vee \text{ is idempotent } \rangle \\ & z = 3 \;\Rightarrow\; (\exists y \mid z = 3 : y = z) \vee (\exists y \mid y = 3 : y = z) \\ = & \langle \text{ (8.14), } y \text{ d.n.o.f. } 3 \rangle \\ & z = 3 \;\Rightarrow\; (\exists y \mid z = 3 : y = z) \vee 3 = z \\ = & \langle \text{ (1.3) } \rangle \\ & z = 3 \;\Rightarrow\; (\exists y \mid z = 3 : y = z) \vee z = 3 \ . \end{array}$$

which is (3.76)(a).

$$\begin{array}{ll} & z = 3 \;\Rightarrow\; (\exists y \mid z = 3 \vee y = 3 : y = z) \\ = & \langle \text{ (3.84)(b) } \rangle \\ & z = 3 \;\Rightarrow\; (\exists y \mid 3 = 3 \vee y = 3 : y = 3) \\ = & \langle \text{ (1.2) } \rangle \\ & z = 3 \;\Rightarrow\; (\exists y \mid true \vee y = 3 : y = 3) \\ = & \langle \text{ (3.29) } \rangle \\ & z = 3 \;\Rightarrow\; (\exists y \mid \ : y = 3) \ . \end{array}$$

which is (9.28).

Assume $z = 3$.

$$\begin{array}{ll} & (\exists y \mid z = 3 \vee y = 3 : y = z) \\ = & \langle \text{ Assumption, } y \text{ d.n.o.f. } z = 3 \rangle \\ & (\exists y \mid 3 = 3 \vee y = 3 : y = 3) \\ = & \langle \text{ (1.2), (3.29) } \rangle \\ & (\exists y \mid \ : y = 3) \ . \end{array}$$

But $\vdash\ 3 = 3 \;\Rightarrow\; (\exists y \mid \ : y = 3)$ and by Modus Ponens, since by (1.2), $\vdash\ 3 = 3$, $\vdash\ (\exists y \mid y = 3)$.

3. (9 points) Which of the following expressions are theorems?

$$(\exists x \mid\ : P) \quad \Rightarrow \quad (\exists x \mid R : P)\ .$$
$$(\forall x \mid\ : P) \quad \Rightarrow \quad (\forall x \mid R : P)\ .$$

If the expression is **a theorem**, give its proof.
If the expression is **not a theorem**, give an interpretation for which it is f.

Answer: The first expression is **not a theorem**. Take $\{0,1\}$ as Universe of Discourse, P to be "$x = 1$", R to be "$x = 0$. Then $(\exists x \mid\ : P)$ is in state t, while $(\exists x \mid R : P)$ is in state f.

The second expression is a theorem. Here are two different proofs.

$$
\begin{array}{cl}
 & (\forall x \mid\ : P) \Rightarrow (\forall x \mid R : P) \\
= & \langle\ (3.60)\ \rangle \\
 & (\forall x \mid\ : P)\ \wedge\ (\forall x \mid R : P) \quad \equiv \quad (\forall x \mid\ : P) \\
= & \langle\ (9.2)\ \rangle \\
 & (\forall x \mid\ : P)\ \wedge\ (\forall x \mid\ : R \Rightarrow P) \quad \equiv \quad (\forall x \mid\ : P) \\
= & \langle\ (8.15)\ \rangle \\
 & (\forall x \mid\ : P \wedge (R \Rightarrow P)) \quad \equiv \quad (\forall x \mid\ : P) \\
= & \langle\ (3.59)\ \rangle \\
 & (\forall x \mid\ : P \wedge (\neg R \vee P)) \quad \equiv \quad (\forall x \mid\ : P) \\
= & \langle\ (3.43)\ \rangle \\
 & (\forall x \mid\ : P) \quad \equiv \quad (\forall x \mid\ : P)\ .
\end{array}
$$

$$
\begin{array}{cl}
 & (\forall x \mid\ : P) \Rightarrow (\forall x \mid R : P) \\
= & \langle\ (3.57)\ \rangle \\
 & (\forall x \mid\ : P) \vee (\forall x \mid R : P) \quad \equiv \quad (\forall x \mid R : P) \\
= & \langle\ (9.5),\ x \text{ d.n.o.f. } \forall x \mid\ : P)\ \rangle \\
 & (\forall x \mid R : (\forall x \mid\ : P) \vee P) \quad \equiv \quad (\forall x \mid R : P) \\
= & \langle\ \text{From (9.13) and (3.60) one obtains} \vdash\ (\forall x \mid\ : P) \vee P) \equiv P\ \rangle \\
 & (\forall x \mid R : P) \quad \equiv \quad (\forall x \mid R : P)\ .
\end{array}
$$

4. (7 points) **Without using (8.23),** prove that,

$$\vdash (+x \mid 0 \le x \le n : 1) = (+y \mid 1 \le y \le n+1 : 1) \ .$$

Your reasons must indicate that you have verified any conditions concerning free occurrences of variables.
Hint: $\vdash 1 = (+y \mid y = x+1 : 1)$.

Answer:

$$
\begin{array}{cl}
 & (+x \mid 0 \le x \le n : 1) \\
= & \langle \text{ Hint, i.e., (8.14), } y \text{ d.n.o.f. } x+1 \ \rangle \\
 & (+x \mid 0 \le x \le n : (+y \mid y = x+1 : 1)) \\
= & \langle \text{ (8.20), } y \text{ d.n.o.f. } 0 \le x \le n \ \rangle \\
 & (+x, y \mid 0 \le x \le n \vee y = x+1 : 1) \\
= & \langle \text{ Algebra } \rangle \\
 & (+y, x \mid 0 \le x \le n \vee x = y-1 : 1) \\
= & \langle \text{ (3.84)(a) } \rangle \\
 & (+y, x \mid 0 \le y-1 \le n \vee x = y-1 : 1) \\
= & \langle \text{ (8.20), } x \text{ d.n.o.f. } y-1 \ \rangle \\
 & (+y \mid 0 \le y-1 \le n : (+x \mid x = y-1 : 1)) \\
= & \langle \text{ (8.14), } x \text{ d.n.o.f. } y-1 \ \rangle \\
 & (+y \mid 0 \le y-1 \le n : 1) \\
= & \langle \text{ Algebra } \rangle \\
 & (+y \mid 1 \le y \le n+1 : 1) \ .
\end{array}
$$

The end