York University

Faculty of Science and Engineering

Math 1200 C

Final Examination

NAME (print): ____
(Family) (Given)

SIGNATURE: ____

STUDENT NUMBER: ____

Instructions:

1. Time allowed: 180 minutes.
2. There are 7 questions on 11 pages.
3. Questions can be solved in more than one way.
4. Make sure that you write clearly and carefully using full English sentences.

Question	Points	Marks
1	10	
2	10	
3	10	
4	20	
5	20	
6	15	
7	15	
BONUS	2	
Total	$100 + 2$	

IF YOU HAVE COMPLETED THE ONLINE COURSE EVALUATION, SIGN THIS STATEMENT TO RECEIVE YOUR BONUS POINTS.

I have completed the online course evaluation.

SIGNATURE: ____

1. (10 points) Prove or disprove, the sum of any four consecutive integers is even.

2. (10 points) Let x be a real number. Prove or disprove, $x = 2$ if and only if $x^3 - x^2 - x = 2$.

3. (10 points)

$$1 \quad 2 \quad 3 \quad \cdots \quad n \quad (n+1) \quad \cdots \quad (2n-2) \quad (2n-1) \quad (2n)$$

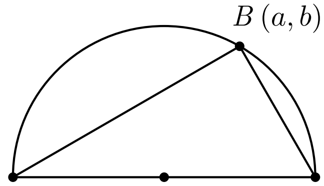
Consider the natural numbers from 1 to $2n$. Pair off these numbers as above, 1 and $(2n)$, 2 and $(2n-1)$, 3 and $(2n-2)$, $\ldots$, n and $(n+1)$, and evaluate the products of the pairs, $1 \times (2n)$, $2 \times (2n-1)$, $3 \times (2n-2)$, $\ldots$, $n \times (n+1)$. Prove that for no value of n are two of these n products equal.

4. (20 points) Consider the statement, the sum of any three consecutive positive perfect cubes is divisible by 9.

   (a) Sum the cubes of 4, 5 and 6 and verify that the resulting number is divisible by 9.

   (b) Prove the statement using mathematical induction.

(c) Prove the statement by considering three cases, depending on the remainder when the smallest number cubed is divided by 3.



5. (20 points)

   (a) Given that $a^2 + b^2 = r^2$, i.e., the point B lies on the circle with center O and radius r, use the slopes of the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ to prove that $\angle ABC$ is a right angle.
   Note: This gives an alternate way of proving that an angle inscribed in a semi-circle is a right angle.

(b) Fix A and C as in the diagram. What can be said about the size of $\angle ABC$ if B lies inside the circle? Justify your answer.

(c) Fix A and C as in the diagram. What can be said about the size of $\angle ABC$ if B lies outside the circle? Justify your answer.

6. (15 points) Read the following excerpt from Stillwell, *Mathematics and its History*, and use it as required to answer the following questions. Note that p and q represent positive integers.

As early as 2000 B.C., the Babylonians could solve pairs of simultaneous equations of the form

$$\begin{aligned} x + y &= p \\ xy &= q \end{aligned}$$

which are equivalent to the quadratic equations

$$x^2 + q = px$$

The original pair was solved by a method that gave the two roots of the quadratic:

$$x, y = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

when both [of these roots] were positive (the Babylonians did not admit negative numbers). The steps in the method were:

(i) Form $\dfrac{x+y}{2}$

(ii) Form $\left(\dfrac{x+y}{2}\right)^2$

(iii) Form $\left(\dfrac{x+y}{2}\right)^2 - xy$

(iv) Form $\sqrt{\left(\dfrac{x+y}{2}\right)^2 - xy} = \dfrac{x-y}{2}$

(v) Find x, y by inspection of the values in (i), (iv)

Of course these steps were not expressed in symbols but only applied to specific numbers. Nevertheless, a general method is implicit in the many specific cases solved.

(a) Why is solving the system

$$\begin{aligned} x + y &= p \\ xy &= q \end{aligned}$$

equivalent to solving the quadratic equation

$$x^2 + q = px\,?$$

Hint: Compare the solution of the system with the two roots of the quadratic.

(b) Solve $x^2 + 3 = 4x$ using the Babylonian method. Clearly indicate each step of the procedure.

7. (15 points) Consider the following statement.

   Let p and q be integers. If both $x^2 + px - q = 0$ and $x^2 + px + q = 0$ have integer solutions then there exist integers a and b such that $a^2 + b^2 = p^2$.

   The following leads to a proof.

   (a) State the quadratic formula and use the quadratic formula to verify that both $x^2 + 5x - 6 = 0$ and $x^2 + 5x + 6 = 0$ have integer solutions.

   (b) Recall that two integers have the *same parity* if both are even or both are odd. Let $m^2 = p^2 + 4q$ and $k^2 = p^2 - 4q$ with p, q, m, k integers. Prove that m and k have the same parity.

   (c) Verify the identity,

$$\left(\frac{m+k}{2}\right)^2 + \left(\frac{m-k}{2}\right)^2 = \frac{m^2+k^2}{2}\,.$$

(d) Prove the statement. Give formulas for the values of a and b.

(e) Verify that in the case $p = 13,\ q = 30$, the formula you obtain in (d) gives the values 12 and 5 for a and b.

The end