

York University

Faculty of Science and Engineering

Math 1200 C

Final Examination

NAME (print): _____
(Family) (Given)

SIGNATURE: _____

STUDENT NUMBER: _____

Instructions:

1. Time allowed: 180 minutes.
2. There are 7 questions on 11 pages.
3. Questions can be solved in more than one way.
4. Make sure that you write clearly and carefully using full English sentences.

Question	Points	Marks
1	10	
2	10	
3	10	
4	20	
5	20	
6	15	
7	15	
BONUS	2	
Total	100 + 2	

IF YOU HAVE COMPLETED THE ONLINE COURSE EVALUATION, SIGN THIS STATEMENT TO RECEIVE YOUR BONUS POINTS.

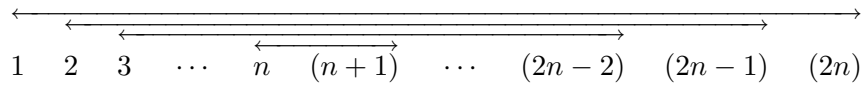
I have completed the online course evaluation.

SIGNATURE: _____

1. (10 points) Prove or disprove, the sum of any four consecutive integers is even.

2. (10 points) Let x be a real number. Prove or disprove, $x = 2$ if and only if $x^3 - x^2 - x = 2$.

3. (10 points)



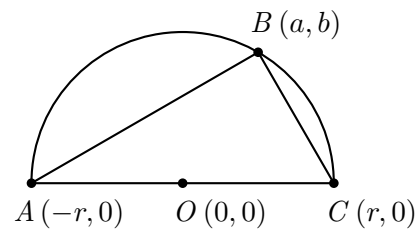
Consider the natural numbers from 1 to $2n$. Pair off these numbers as above, 1 and $(2n)$, 2 and $(2n-1)$, 3 and $(2n-2)$, \dots , n and $(n+1)$, and evaluate the products of the pairs, $1 \times (2n)$, $2 \times (2n-1)$, $3 \times (2n-2)$, \dots , $n \times (n+1)$. Prove that for no value of n are two of these n products equal.

4. (20 points) Consider the statement, the sum of any three consecutive positive perfect cubes is divisible by 9.

(a) Sum the cubes of 4, 5 and 6 and verify that the resulting number is divisible by 9.

(b) Prove the statement using mathematical induction.

- (c) Prove the statement by considering three cases, depending on the remainder when the smallest number cubed is divided by 3.



5. (20 points)

- (a) Given that $a^2 + b^2 = r^2$, i.e., the point B lies on the circle with center O and radius r , use the slopes of the lines \overleftrightarrow{AB} and \overleftrightarrow{BC} to prove that $\angle ABC$ is a right angle.

Note: This gives an alternate way of proving that an angle inscribed in a semi-circle is a right angle.

(b) Fix A and C as in the diagram. What can be said about the size of $\angle ABC$ if B lies inside the circle? Justify your answer.

(c) Fix A and C as in the diagram. What can be said about the size of $\angle ABC$ if B lies outside the circle? Justify your answer.

6. (15 points) Read the following excerpt from Stillwell, *Mathematics and its History*, and use it as required to answer the following questions. Note that p and q represent positive integers.

As early as 2000 B.C., the Babylonians could solve pairs of simultaneous equations of the form

$$\begin{aligned}x + y &= p \\ xy &= q\end{aligned}$$

which are equivalent to the quadratic equations

$$x^2 + q = px$$

The original pair was solved by a method that gave the two roots of the quadratic:

$$x, y = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

when both [of these roots] were positive (the Babylonians did not admit negative numbers). The steps in the method were:

- (i) Form $\frac{x + y}{2}$
- (ii) Form $\left(\frac{x + y}{2}\right)^2$
- (iii) Form $\left(\frac{x + y}{2}\right)^2 - xy$
- (iv) Form $\sqrt{\left(\frac{x + y}{2}\right)^2 - xy} = \frac{x - y}{2}$
- (v) Find x, y by inspection of the values in (i), (iv)

Of course these steps were not expressed in symbols but only applied to specific numbers. Nevertheless, a general method is implicit in the many specific cases solved.

(a) Why is solving the system

$$\begin{aligned}x + y &= p \\xy &= q\end{aligned}$$

equivalent to solving the quadratic equation

$$x^2 + q = px?$$

Hint: Compare the solution of the system with the two roots of the quadratic.

(b) Solve $x^2 + 3 = 4x$ using the Babylonian method. Clearly indicate each step of the procedure.

7. (15 points) Consider the following statement.

Let p and q be integers. If both $x^2 + px - q = 0$ and $x^2 + px + q = 0$ have integer solutions then there exist integers a and b such that $a^2 + b^2 = p^2$.

The following leads to a proof.

(a) State the quadratic formula and use the quadratic formula to verify that both $x^2 + 5x - 6 = 0$ and $x^2 + 5x + 6 = 0$ have integer solutions.

(b) Recall that two integers have the *same parity* if both are even or both are odd. Let $m^2 = p^2 + 4q$ and $k^2 = p^2 - 4q$ with p, q, m, k integers. Prove that m and k have the same parity.

(c) Verify the identity,

$$\left(\frac{m+k}{2}\right)^2 + \left(\frac{m-k}{2}\right)^2 = \frac{m^2+k^2}{2}.$$

(d) Prove the statement. Give formulas for the values of a and b .

(e) Verify that in the case $p = 13$, $q = 30$, the formula you obtain in (d) gives the values 12 and 5 for a and b .

The end