4

Pictorial proofs

Have you ever worked through a proof, understood and confirmed each step, yet still not believed the theorem? You realize *that* the theorem is true, but not *why* it is true.

To see the same contrast in a familiar example, imagine learning that your child has a fever and hearing the temperature in Fahrenheit or Celsius degrees, whichever is less familiar. In my everyday experience, temperatures are mostly in Fahrenheit. When I hear about a temperature of 40°C, I therefore react in two stages:

1. I convert 40° C to Fahrenheit: $40 \times 1.8 + 32 = 104$.
2. I react: "Wow, 104° F. That's dangerous! Get thee to a doctor!"

The Celsius temperature, although symbolically equivalent to the Fahrenheit temperature, elicits no reaction. My danger sense activates only after the temperature conversion connects the temperature to my experience.

A symbolic description, whether a proof or an unfamiliar temperature, is unconvincing compared to an argument that speaks to our perceptual system. The reason lies in how our brains acquired the capacity for symbolic reasoning. (See *Evolving Brains* [2] for an illustrated, scholarly history of the brain.) Symbolic, sequential reasoning requires language, which has evolved for only 10^5 yr. Although 10^5 yr spans many human lifetimes, it is an evolutionary eyeblink. In particular, it is short compared to the time span over which our perceptual hardware has evolved: For several hundred million years, organisms have refined their capacities for hearing, smelling, tasting, touching, and seeing.

Evolution has worked 1000 times longer on our perceptual abilities than on our symbolic-reasoning abilities. Compared to our perceptual hardware, our symbolic, sequential hardware is an ill-developed latecomer. Not surprisingly, our perceptual abilities far surpass our symbolic abilities. Even an apparently high-level symbolic activity such as playing grandmaster chess uses mostly perceptual hardware [16]. *Seeing* an idea conveys to us a depth of understanding that a symbolic description of it cannot easily match.

Problem 4.1 Computers versus people
At tasks like expanding $(x+2y)^{50}$, computers are much faster than people. At tasks like recognizing faces or smells, even young children are much faster than current computers. How do you explain these contrasts?

Problem 4.2 Linguistic evidence for the importance of perception
In your favorite language(s), think of the many sensory synonyms for understanding (for example, grasping).

4.1 Adding odd numbers

To illustrate the value of pictures, let's find the sum of the first n odd numbers (also the subject of Problem 2.25):

$$S_n = \underbrace{1+3+5+\cdots+(2n-1)}_{n \text{ terms}}. \tag{4.1}$$

Easy cases such as $n = 1$, 2, or 3 lead to the conjecture that $S_n = n^2$. But how can the conjecture be proved? The standard symbolic method is proof by induction:

1. Verify that $S_n = n^2$ for the *base case* $n = 1$. In that case, S_1 is 1, as is n^2, so the base case is verified.
2. Make the *induction hypothesis:* Assume that $S_m = m^2$ for m less than or equal to a maximum value n. For this proof, the following, weaker induction hypothesis is sufficient:

$$\sum_{1}^{n}(2k-1) = n^2. \tag{4.2}$$

In other words, we assume the theorem only in the case that $m = n$.

3. Perform the *induction step:* Use the induction hypothesis to show that $S_{n+1} = (n+1)^2$. The sum S_{n+1} splits into two pieces:

$$S_{n+1} = \sum_{1}^{n+1}(2k-1) = (2n+1) + \sum_{1}^{n}(2k-1). \tag{4.3}$$

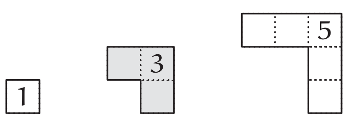
Thanks to the induction hypothesis, the sum on the right is n^2. Thus

$$S_{n+1} = (2n+1) + n^2, \tag{4.4}$$

which is $(n+1)^2$; and the theorem is proved.

Although these steps prove the theorem, *why* the sum S_n ends up as n^2 still feels elusive.

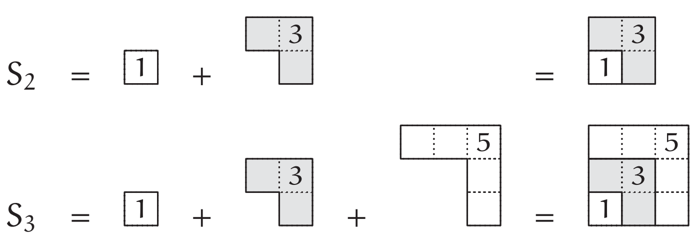
That missing understanding—the kind of gestalt insight described by Wertheimer [48]—requires a pictorial proof. Start by drawing each odd number as an L-shaped puzzle piece:



(4.5)

▶ *How do these pieces fit together?*

Then compute S_n by fitting together the puzzle pieces as follows:



(4.6)

Each successive odd number—each piece—extends the square by 1 unit in height and width, so the n terms build an $n \times n$ square. [Or is it an $(n-1) \times (n-1)$ square?] Therefore, their sum is n^2. After grasping this pictorial proof, you cannot forget why adding up the first n odd numbers produces n^2.

Problem 4.3 Triangular numbers
Draw a picture or pictures to show that

$$1+2+3+\cdots+n+\cdots+3+2+1=n^2. \tag{4.7}$$

Then show that

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}. \tag{4.8}$$

Problem 4.4 Three dimensions
Draw a picture to show that

$$\sum_0^n (3k^2+3k+1)=(n+1)^3. \tag{4.9}$$

Give pictorial explanations for the 1 in the summand $3k^2+3k+1$; for the 3 and the k^2 in $3k^2$; and for the 3 and the k in $3k$.

4.2 Arithmetic and geometric means

The next pictorial proof starts with two nonnegative numbers—for example, 3 and 4—and compares the following two averages:

$$\text{arithmetic mean} \equiv \frac{3+4}{2} = 3.5; \tag{4.10}$$

$$\text{geometric mean} \equiv \sqrt{3\times 4} \approx 3.464. \tag{4.11}$$

Try another pair of numbers—for example, 1 and 2. The arithmetic mean is 1.5; the geometric mean is $\sqrt{2}\approx 1.414$. For both pairs, the geometric mean is smaller than the arithmetic mean. This pattern is general; it is the famous arithmetic-mean–geometric-mean (AM–GM) inequality [18]:

$$\underbrace{\frac{a+b}{2}}_{\text{AM}} \geqslant \underbrace{\sqrt{ab}}_{\text{GM}}. \tag{4.12}$$

(The inequality requires that $a, b \geqslant 0$.)

Problem 4.5 More numerical examples
Test the AM–GM inequality using varied numerical examples. What do you notice when a and b are close to each other? Can you formalize the pattern? (See also Problem 4.16.)

4.2.1 Symbolic proof

The AM–GM inequality has a pictorial and a symbolic proof. The symbolic proof begins with $(a-b)^2$—a surprising choice because the inequality contains $a+b$ rather than $a-b$. The second odd choice is to form $(a-b)^2$. It is nonnegative, so $a^2-2ab+b^2 \geqslant 0$. Now magically decide to add $4ab$ to both sides. The result is

$$\underbrace{a^2+2ab+b^2}_{(a+b)^2} \geqslant 4ab. \tag{4.13}$$

The left side is $(a+b)^2$, so $a+b \geqslant 2\sqrt{ab}$ and

$$\frac{a+b}{2} \geqslant \sqrt{ab}. \tag{4.14}$$

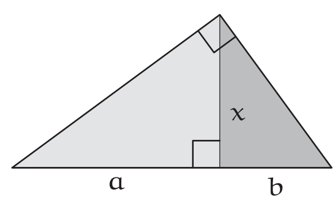
Although each step is simple, the whole chain seems like magic and leaves the *why* mysterious. If the algebra had ended with $(a+b)/4 \geqslant \sqrt{ab}$, it would not look obviously wrong. In contrast, a convincing proof would leave us feeling that the inequality cannot help but be true.

4.2.2 Pictorial proof

This satisfaction is provided by a pictorial proof.

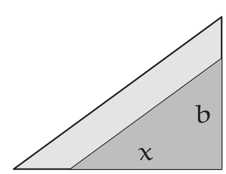
▶ *What is pictorial, or geometric, about the geometric mean?*

A geometric picture for the geometric mean starts with a right triangle. Lay it with its hypotenuse horizontal; then cut it with the altitude x into the light and dark subtriangles. The hypotenuse splits into two lengths a and b, and the altitude x is their geometric mean $\sqrt{ab}$.



▶ *Why is the altitude x equal to $\sqrt{ab}$?*

To show that $x=\sqrt{ab}$, compare the small, dark triangle to the large, light triangle by rotating the small triangle and laying it on the large triangle. The two triangles are similar! Therefore, their aspect ratios (the ratio of the short to the long side) are identical. In symbols, $x/a = b/x$: The altitude x is therefore the geometric mean $\sqrt{ab}$.

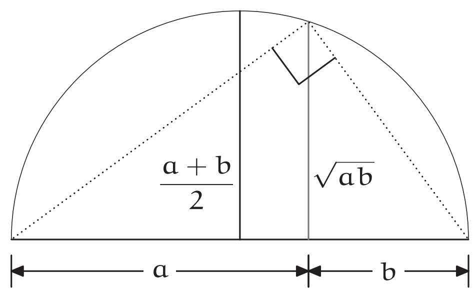


The uncut right triangle represents the geometric-mean portion of the AM–GM inequality. The arithmetic mean $(a+b)/2$ also has a picture, as one-half of the hypotenuse. Thus, the inequality claims that

$$\frac{\text{hypotenuse}}{2} \geqslant \text{altitude}. \tag{4.15}$$

Alas, this claim is not pictorially obvious.

Can you find an alternative geometric interpretation of the arithmetic mean that makes the AM–GM inequality pictorially obvious?

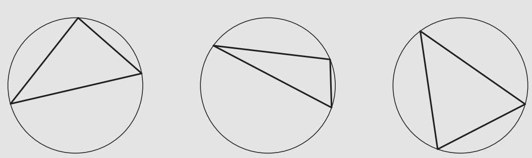


The arithmetic mean is also the radius of a circle with diameter $a+b$. Therefore, circumscribe a semicircle around the triangle, matching the circle's diameter with the hypotenuse $a+b$ (Problem 4.7). The altitude cannot exceed the radius; therefore,

$$\frac{a+b}{2} \geqslant \sqrt{ab}. \tag{4.16}$$

Furthermore, the two sides are equal only when the altitude of the triangle is also a radius of the semicircle—namely when $a = b$. The picture therefore contains the inequality and its equality condition in one easy-to-grasp object. (An alternative pictorial proof of the AM–GM inequality is developed in Problem 4.33.)

Problem 4.6 Circumscribing a circle around a triangle
Here are a few examples showing a circle circumscribed around a triangle.



Draw a picture to show that the circle is uniquely determined by the triangle.

Problem 4.7 Finding the right semicircle
A triangle uniquely determines its circumscribing circle (Problem 4.6). However, the circle's diameter might not align with a side of the triangle. Can a semicircle always be circumscribed around a right triangle while aligning the circle's diameter along the hypotenuse?

Problem 4.8 Geometric mean of three numbers

For three nonnegative numbers, the AM–GM inequality is

$$\frac{a+b+c}{3} \geqslant (abc)^{1/3}. \tag{4.17}$$

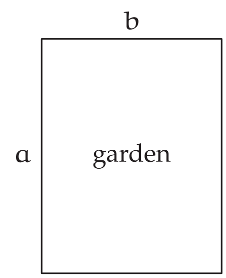
Why is this inequality, in contrast to its two-number cousin, unlikely to have a geometric proof? (If you find a proof, let me know.)

4.2.3 Applications

Arithmetic and geometric means have wide mathematical application. The first application is a problem more often solved with derivatives: Fold a fixed length of fence into a rectangle enclosing the largest garden.

▶ *What shape of rectangle maximizes the area?*

The problem involves two quantities: a perimeter that is fixed and an area to maximize. If the perimeter is related to the arithmetic mean and the area to the geometric mean, then the AM–GM inequality might help maximize the area. The perimeter $P = 2(a+b)$ is four times the arithmetic mean, and the area $A = ab$ is the square of the geometric mean. Therefore, from the AM–GM inequality,



$$\underbrace{\frac{P}{4}}_{\text{AM}} \geqslant \underbrace{\sqrt{A}}_{\text{GM}} \tag{4.18}$$

with equality when $a = b$. The left side is fixed by the amount of fence. Thus the right side, which varies depending on a and b, has a maximum of $P/4$ when $a = b$. The maximal-area rectangle is a square.

Problem 4.9 Direct pictorial proof

The AM–GM reasoning for the maximal rectangular garden is indirect pictorial reasoning. It is symbolic reasoning built upon the pictorial proof for the AM–GM inequality. Can you draw a picture to show directly that the square is the optimal shape?

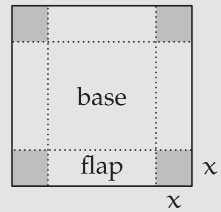
Problem 4.10 Three-part product

Find the maximum value of $f(x) = x^2(1-2x)$ for $x \geqslant 0$, without using calculus. Sketch $f(x)$ to confirm your answer.

Problem 4.11 Unrestricted maximal area

If the garden need not be rectangular, what is the maximal-area shape?

Problem 4.12 Volume maximization



Build an open-topped box as follows: Start with a unit square, cut out four identical corners, and fold in the flaps. The box has volume $V = x(1-2x)^2$, where x is the side length of a corner cutout. What choice of x maximizes the volume of the box?

Here is a plausible analysis modeled on the analysis of the rectangular garden. Set $a = x$, $b = 1-2x$, and $c = 1-2x$. Then abc is the volume V, and $V^{1/3} = \sqrt[3]{abc}$ is the geometric mean (Problem 4.8). Because the geometric mean never exceeds the arithmetic mean and because the two means are equal when $a = b = c$, the maximum volume is attained when $x = 1-2x$. Therefore, choosing $x = 1/3$ should maximize the volume of the box.

Now show that this choice is wrong by graphing $V(x)$ or setting $dV/dx = 0$; explain what is wrong with the preceding reasoning; and make a correct version.

Problem 4.13 Trigonometric minimum

Find the minimum value of

$$\frac{9x^2 \sin^2 x + 4}{x \sin x} \tag{4.19}$$

in the region $x \in (0, \pi)$.

Problem 4.14 Trigonometric maximum

In the region $t \in [0, \pi/2]$, maximize $\sin 2t$ or, equivalently, $2 \sin t \cos t$.

The second application of arithmetic and geometric means is a modern, amazingly rapid method for computing π [5, 6]. Ancient methods for computing π included calculating the perimeter of many-sided regular polygons and provided a few decimal places of accuracy.

Recent computations have used Leibniz's arctangent series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots. \tag{4.20}$$

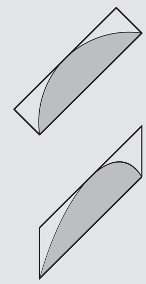
Imagine that you want to compute π to 10^9 digits, perhaps to test the hardware of a new supercomputer or to study whether the digits of π are random (a theme in Carl Sagan's novel *Contact* [40]). Setting $x = 1$ in the Leibniz series produces $\pi/4$, but the series converges extremely slowly. Obtaining 10^9 digits requires roughly 10^{10^9} terms—far more terms than atoms in the universe.

the largest supercomputer. Pictorial reasoning, therefore, taps the mind's vast computational power. It makes us more intelligent by helping us understand and see large ideas at a glance.

For extensive and enjoyable collections of picture proofs, see the works of Nelsen [31, 32]. Here are further problems to develop pictorial reasoning.

Problem 4.33 Another picture for the AM–GM inequality
Sketch $y = \ln x$ to show that the arithmetic mean of a and b is always greater than or equal to their geometric mean, with equality when $a = b$.

Problem 4.34 Archimedes' formula for the area of a parabola



Archimedes showed (long before calculus!) that the closed parabola encloses two-thirds of its circumscribing rectangle. Prove this result by integration.

Show that the closed parabola also encloses two-thirds of the circumscribing parallelogram with vertical sides. These pictorial recipes are useful when approximating functions (for example, in Problem 4.32).

Problem 4.35 Ancient picture for the area of a circle
The ancient Greeks knew that the circumference of a circle with radius r was $2\pi r$. They then used the following picture to show that its area is πr^2. Can you reconstruct the argument?

Problem 4.36 Volume of a sphere
Extend the argument of Problem 4.35 to find the volume of a sphere of radius r, given that its surface area is $4\pi r^2$. Illustrate the argument with a sketch.

Problem 4.37 A famous sum

Use pictorial reasoning to approximate the famous Basel sum $\sum_{1}^{\infty} n^{-2}$.

Problem 4.38 Newton–Raphson method
In general, solving $f(t) = 0$ requires approximations. One method is to start with a guess t_0 and to improve it iteratively using the Newton–Raphson method

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}, \tag{4.52}$$

where $f'(t_n)$ is the derivative df/dt evaluated at $t = t_n$. Draw a picture to justify this recipe; then use the recipe to estimate $\sqrt{2}$. (Then try Problem 4.17.)