York University

Faculty of Science and Engineering

Math 1200

Final Examination

NAME (print): ____

(Family) (Given)

SIGNATURE: ____

STUDENT NUMBER: ____

SECTION LETTER:

[ ]

(A: Brettler, Tuesday) (B: Zabrocki, Monday) (C: Zabrocki, Thursday)

Instructions:

1. Time allowed: 180 minutes.
2. There are 6 questions on 14 pages. The last page is blank.
3. Questions can be solved in more than one way.
4. Make sure that you write clearly and carefully using full English sentences.

Question	**Points**	**Marks**
1	15	
2	15	
3	15	
4	15	
5	20	
6	20	
BONUS	2	
Total	$100 + 2$	

IF YOU HAVE COMPLETED THE ONLINE COURSE EVALUATION, SIGN THIS STATEMENT TO RECEIVE YOUR BONUS POINTS.

I have completed the online course evaluation.

SIGNATURE: ____

1. (15 points)

   (a) Prove that the sum of the squares of four consecutive even integers is divisible by 8.

(b) Is the sum of the squares of four consecutive odd integers ever divisible by 8? Justify your answer.

2. (15 points) For $n \geq 1$, let

$$f(n) = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)}\,.$$

(a) Evaluate $f(n)$ for $n = 1, 2, 3, 4$. Write your answer as an ordinary fraction.

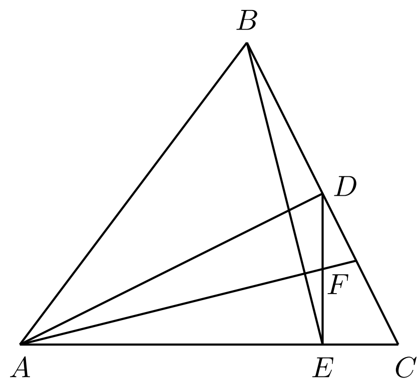
(b) Write down a formula for $f(n)$ and use mathematical induction to prove that it is correct.

3. (15 points) Recall the following.

Let (x_1, y_1) and (x_2, y_2) be two points in the Cartesian plane.

- The distance between them is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The midpoint of the segment between them is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.
- The slope of the line through them is $\dfrac{y_2 - y_1}{x_2 - x_1}$.

Two lines are perpendicular when the product of their slopes is -1.
The foot of the perpendicular from the point (x_1, y_1) to the horizontal axis has coordinates $(x_1, 0)$.



In triangle ABC, $AB = AC$, D is the midpoint of BC, E is the foot of the perpendicular drawn from D to AC, and F is the midpoint of DE. The goal is to prove that AF is perpendicular to BE.

Let $A = (0, 0)$, $B = (4a, 4b)$, $C = (4c, 0)$.

(a) Explain why $a^2 + b^2 = c^2$.

(b) Find the slopes of the line $\overleftrightarrow{AF}$ and of the line $\overleftrightarrow{BE}$.

(c) Explain why $\overleftrightarrow{AF} \perp \overleftrightarrow{BE}$.

4. (15 points) For $n \geq 0$, define numbers $A_{n,k}$ by $A_{n,0} = A_{n,n} = n + 2$ and if $1 \leq k \leq n$, by $A_{n,k} = A_{n-1,k-1} + A_{n-1,k}$.

   (a) Evaluate these numbers for $n = 0, 1, 2, 3, 4, 5$ (or more) and arrange them in a triangle.

   (b) Compare your triangle in (a) to Pascal's triangle. Use ordinary language to make the comparison.

   (c) Is $A_{n,k} = C_{n+2,k+1}$? Justify your answer.

5. (20 points) Consider the following question from Larson, *Problem-Solving Through Problems.*

Which is larger $\sqrt[3]{60}$ or $2+\sqrt[3]{7}$? (Cubing each number leads to complications that are not easily resolved. Consider instead the more general problem: Which is larger, $\sqrt[3]{4(x+y)}$ or $\sqrt[3]{x}+\sqrt[3]{y}$, where $x, y \geq 0$? Take $x=a^3, y=b^3$.)

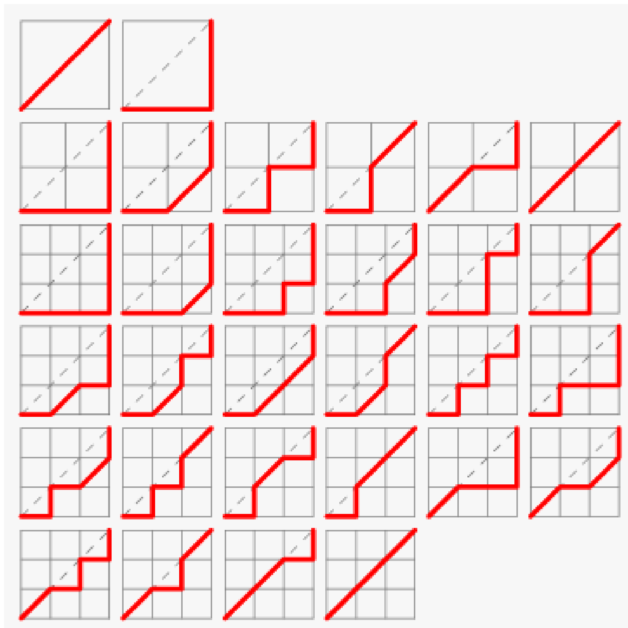
(a) In general, for $x, y \geq 0$, $\sqrt[3]{4(x+y)} \geq \sqrt[3]{x}+\sqrt[3]{y}$. Which values of x and y can be used to establish which of $\sqrt[3]{60}$ and $2+\sqrt[3]{7}$ is bigger? Which is bigger? Explain why.

(b) Following Larson's suggestion, let $x=a^3$ and $y=b^3$. Show that for $a, b \geq 0$, $4(a^3+b^3) \geq (a+b)^3$.
Hint: First verify that $4(a^3+b^3)-(a+b)^3=3(a+b)(a-b)^2$.

(c) Explain how using (b) you can conclude that for $x, y \geq 0$, $\sqrt[3]{4(x+y)} \geq \sqrt[3]{x}+\sqrt[3]{y}$.

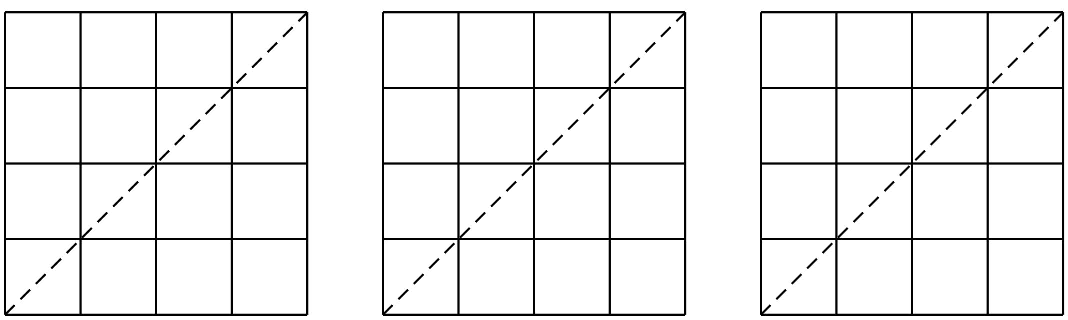
(d) For $u, v \geq 0$ use a similar argument to compare $\sqrt{2(u+v)}$ and $\sqrt{u} + \sqrt{v}$.

6. (20 points) Schröder paths are defined to be paths from the soutwest corner $(0,0)$ of an $n \times n$ grid to the northeast corner (n,n), using only single steps north, northeast, or east, that do not rise above the SW-NE diagonal. The complete set of paths for $n=1, n=2, n=3$ is shown below. Let S_n denote the number of Schröder paths in an $n \times n$ grid. Just count to obtain $S_1 = 2$, $S_2 = 6$, $S_3 = 22$.



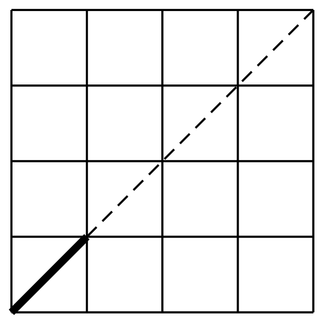
Use R to represent movement of one unit to the right (i.e., east), U to represent movement of one unit up (i.e., north), and D to represent movement one unit diagonally up and to the right (i.e., northeast).

(a) Consider 4×4 grids.
Which of RDRUUD, DRRDUU and DRUURRU represent Schröder paths? For those which do not, explain in one sentence why not.
Hint: You may want to draw them on the grids below.

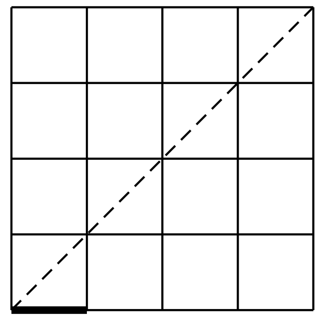


(b) Explain why any Schröder path in a 4×4 grid must begin D or R.

(c) There are $S_3 = 22$ **Schröder paths starting with D**. Explain why.



(d) **Consider Schröder paths starting with R**.



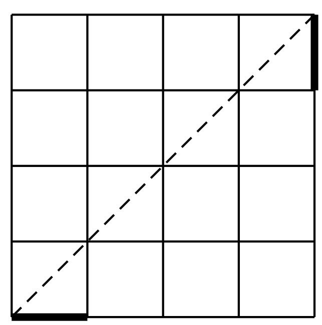
A Schröder path starting with R must eventually hit the diagonal, i.e., reach the point $(1,1)$ or $(2,2)$ or $(3,3)$ or $(4,4)$.

The last move before a path **first** hits the diagonal must be U. Explain why.

Consider four cases.

- The path first hits the diagonal at $(4,4)$:

  What can it look like? The first move is R and the last is U. The completion of the path is a path from (1,0) to (4,3).

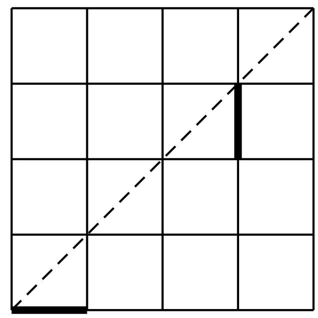
  

  It cannot cross the line between (1,0) to (4,3). Explain why not.

  There are S_3 Schröder paths starting with R which first hit the diagonal at $(4,4)$. Explain.

- The path first hits the diagonal at $(3, 3)$

  What can it look like? The first move is R and it contains U from $(3, 2)$ to $(3, 3)$.



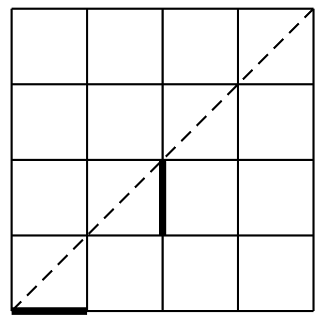
  There are S_2 possible ways to get from $(1, 0)$ to $(3, 2)$. Explain.

  There are S_1 ways to get from $(3, 3)$ to $(4, 4)$. Explain.

  In all there are $S_2 \cdot S_1$ Schröder paths starting with R which first hit the diagonal at $(3, 3)$.

- The path first hits the diagonal at $(2, 2)$

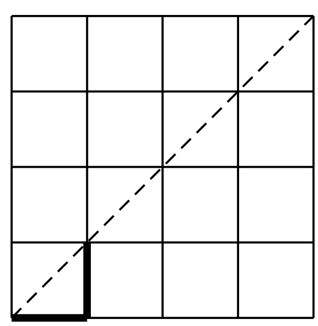
  What can it look like? The first move is R and it contains U from $(2, 1)$ to $(2, 2)$.



  There are $S_1 \cdot S_2$ ways to complete the path, i.e., there are $S_1 \cdot S_2$ Schröder paths starting with R which first hit the diagonal at $(2, 2)$. Explain.

- The path first hits the diagonal at $(1, 1)$

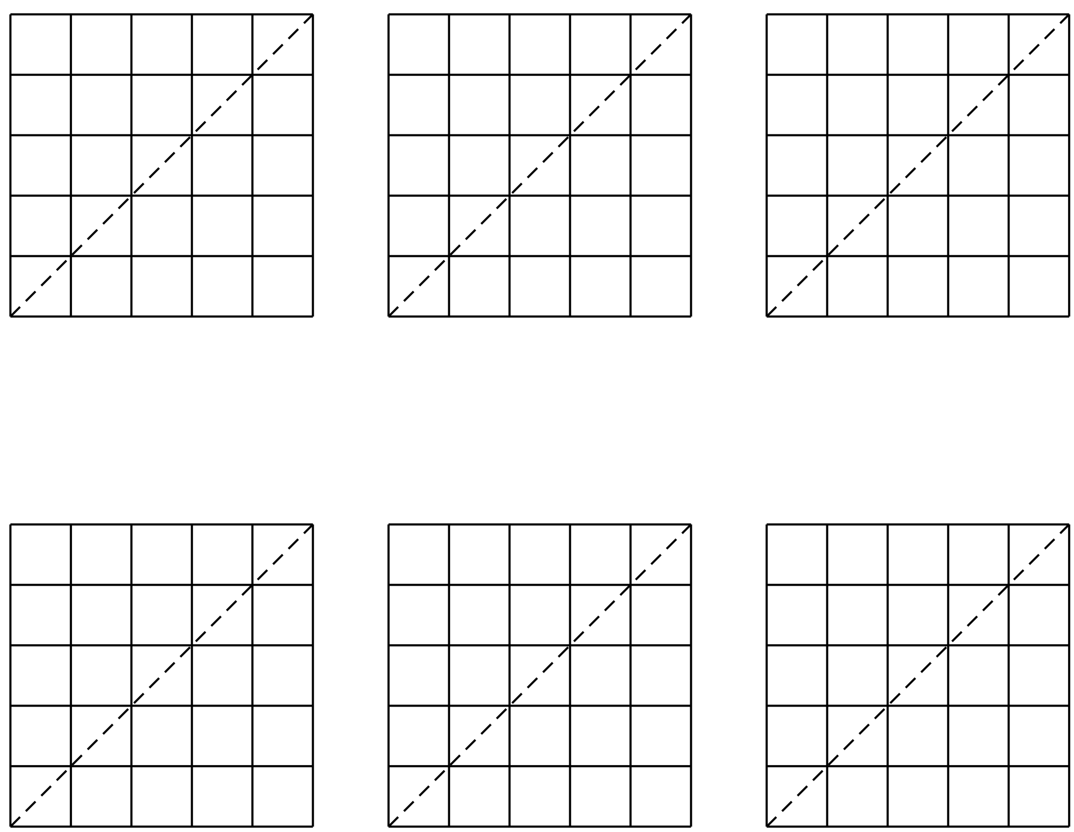
  What can it look like? The first move is R and the second is U.



  There are S_3 ways to complete the path, i.e., there are S_3 Schröder paths starting with R which first hit the diagonal at $(1, 1)$. Explain.

- In all, how many Schröder paths are there in a 4×4 grid?

- Evaluate S_5, the number of Schröder paths in a 5×5 grid. You may want to use these 5×5 grids below to help keep track of the cases you consider.



The end