Chapter 11 Fixes

Set comprehension expressions are constructed in a similar fashion to quantification expressions. Both have dummies, ranges and bodies. For set comprehensions we have dummy renaming as well as Leibniz rules for ranges and bodies.

Theorem, Dummy Renaming: Let R be a boolean expression, E an expression, x a variable. Let w be a variable of the same type as x, w d.n.o.f. in R, E. Then

$$\vdash \{x \mid R : E\} \;=\; \{w \mid R[x := w] : E[x := w]\} \;.$$

Let z be a fresh variable of the same type as x. By (11.4) and (9.16) it suffices to prove

$$\vdash z \in \{x \mid R : E\} \;\equiv\; z \in \{w \mid R[x := w] : E[x := w]\} \;.$$

$$\begin{array}{cl} & z \in \{w \mid R[x := w] : E[x := w]\} \\ = & \langle \text{ (11.3), } w \text{ d.n.o.f. in } z \ \rangle \\ & (\exists w \mid R[x := w] : z = E[x := w]) \\ = & \langle \text{ (8.21), } x \text{ d.n.o.f. in } R[x := w], z = E[x := w] \ \rangle \\ & (\exists x \mid R[x := w][w := x] : z = E[x := w][w := x]) \\ = & \langle \text{ Definition of contextual substitution } \rangle \\ & (\exists x \mid R : z = E) \\ = & \langle \text{ (11.3), } x \text{ d.n.o.f. in } z \ \rangle \\ & z \in \{x \mid R : E\} \;. \end{array}$$

Recall that P is an A–theorem, denoted $A \vdash P$, provided P can be proved in the extended system obtained by adding A as a temporary axiom. By the Deduction Theorem and Strong Modus Ponens, this is the same as the assertion, $\vdash A \Rightarrow P$. Recall that if $\vdash P$, then $A \vdash P$, so that all genuine theorems are A–theorems.

Metatheorem, Leibniz for Ranges: If $P = Q$ is an A–theorem and there are no free occurrences of x in A, then $\{x \mid R[z := P] : E\} \;=\; \{x \mid R[z := Q] : E\}$ is an A–theorem.

Let w be a fresh variable of the same type as x. As above, it suffices to prove that given $A \vdash P = Q$,

$$A \vdash w \in \{x \mid R[z := P] : E\} \;\equiv\; w \in \{x \mid R[z := Q] : E\} \;.$$

$$\begin{array}{cl} & w \in \{x \mid R[z := P] : E\} \\ = & \langle \text{ (11.3), } x \text{ d.n.o.f. in } w \ \rangle \\ & (\exists x \mid R[z := P] : w = E) \\ = & \langle \text{ New Leibniz for ranges, } x \text{ d.n.o.f. in } A \ \rangle \\ & (\exists x \mid R[z := Q] : w = E) \\ = & \langle \text{ (11.3), } x \text{ d.n.o.f. in } w, E \ \rangle \\ & w \in \{x \mid R[z := Q] : E\} \;. \end{array}$$

Metatheorem, Leibniz for Bodies: If $R \;\Rightarrow\; (P = Q)$ is an A–theorem and there are no free occurrences of x in A, then $\{x \,|\, R : E[z := P]\} \;=\; \{x \,|\, R : E[z := Q]\}$ is an A–theorem.

The proof is identical in form to the proof given above. Let w be a fresh variable of the same type as x. It suffices to prove that given $A \vdash\; R \Rightarrow (P = Q)$,

$$A \vdash\;\; w \in \{x \,|\, R : E[z := P]\} \;\;\equiv\;\; w \in \{x \,|\, R : E[z := Q]\}\;.$$

$$\begin{array}{cl}
 & w \in \{x \,|\, R : E[z := P]\} \\
= & \langle\ (11.3),\ x \text{ d.n.o.f. in } w\ \rangle \\
 & (\exists x \,|\, R : w = E[z := P]) \\
= & \langle\ \text{Definition of contextual substitution}\ \rangle \\
 & (\exists x \,|\, R : (w = E)[z := P]) \\
= & \langle\ \text{New Leibniz for bodies, } x \text{ d.n.o.f. in } A\ \rangle \\
 & (\exists x \,|\, R : (w = E)[z := Q]) \\
= & \langle\ \text{Definition of contextual substitution}\ \rangle \\
 & (\exists x \,|\, R : w = E[z := Q]) \\
= & \langle\ (11.3),\ x \text{ d.n.o.f. in } w\ \rangle \\
 & w \in \{x \,|\, R : E[z := Q]\}\;.
\end{array}$$