Chapter 14 Fixes

Definition 14 R: We say "a set expression ρ is a *relation expression*" if, with x, y having no free occurrences in ρ,

$$\vdash\ z \in \rho \ \Rightarrow\ (\exists x, y \mid\ : z = \langle x, y\rangle\ )\ .$$

By (3.60), an alternate formulation is

$$\vdash\ (z \in \rho)\ \equiv\ (z \in \rho)\ \wedge\ (\exists x, y \mid\ : z = \langle x, y\rangle\ )\ .$$

Theorem 14 RP: For ρ a relation expression and provided x, y, z d.n.o.f. in ρ, x, y d.n.o.f. in P,

$$\vdash\ (\forall z \mid z \in \rho : P\ )\ \equiv\ (\forall x, y \mid \langle x, y\rangle \in \rho : P\,[z := \langle x, y\rangle]\ )\ .$$

This result can be understood as change of dummy. The proof is suggested by the textbook's treatment of (8.22).

$$
\begin{aligned}
& (\forall x, y \mid \langle x, y\rangle \in \rho : P\,[z := \langle x, y\rangle]\ ) \\
= \quad & \langle\ \text{(9.2) for multiple quantification}\ \rangle \\
& (\forall x, y \mid\ : \langle x, y\rangle \in \rho \Rightarrow P\,[z := \langle x, y\rangle]\ ) \\
= \quad & \langle\ \text{(8.14), } z \text{ d.n.o.f. in } \langle x, y\rangle\ \rangle \\
& (\forall x, y \mid\ : (\forall z \mid z = \langle x, y\rangle : z \in \rho \Rightarrow P\ )\ ) \\
= \quad & \langle\ \text{(9.4)(a)}\ \rangle \\
& (\forall x, y \mid\ : (\forall z \mid z \in \rho : z = \langle x, y\rangle \Rightarrow P\ )\ ) \\
= \quad & \langle\ \text{(8.19) for multiple quantification, } x,\ y \text{ d.n.o.f. in } z \in \rho\ \rangle \\
& (\forall z \mid z \in \rho : (\forall x, y \mid\ : z = \langle x, y\rangle \Rightarrow P\ )\ ) \\
= \quad & \langle\ \text{(9.2) for multiple quantification}\ \rangle \\
& (\forall z \mid z \in \rho : (\forall x, y \mid z = \langle x, y\rangle : P\ )\ ) \\
= \quad & \langle\ \text{(9.6) for multiple quantification, } x,\ y \text{ d.n.o.f. in } P\ \rangle \\
& (\forall z \mid z \in \rho : P\ \vee\ (\forall x, y \mid\ : \neg(z = \langle x, y\rangle))\ ) \\
= \quad & \langle\ \text{(9.18) for multiple quantification}\ \rangle \\
& (\forall z \mid z \in \rho : \neg(\exists x, y \mid\ : z = \langle x, y\rangle)\ \vee\ P\ ) \\
= \quad & \langle\ \text{(3.59)}\ \rangle \\
& (\forall z \mid z \in \rho : (\exists x, y \mid\ : z = \langle x, y\rangle)\ \Rightarrow\ P\ ) \\
= \quad & \langle\ \text{(9.4)(a)}\ \rangle \\
& (\forall z \mid (z \in \rho)\ \wedge\ (\exists x, y \mid\ : z = \langle x, y\rangle) : P\ ) \\
= \quad & \langle\ \text{14 R}\ \rangle \\
& (\forall z \mid z \in \rho : P\ )
\end{aligned}
$$

Theorem 14 RE: For ρ, τ relation expressions and provided x, y d.n.o.f. in ρ or τ,

$$\vdash \ \rho = \tau \quad \equiv \quad (\forall x, y| : \langle x, y\rangle \in \rho \ \equiv \langle x, y\rangle \in \tau) \ .$$

This is a straightforward proof by Mutual implication.
For necessity,

$$\begin{aligned}
& \rho = \tau \quad \Rightarrow \quad (\forall x, y \,| \ : \langle x, y\rangle \in \rho \ \equiv \langle x, y\rangle \in \tau) \\
= \ & \langle \ (3.84)(b) \ \rangle \\
& \rho = \tau \quad \Rightarrow \quad (\forall x, y \,| \ : \langle x, y\rangle \in \rho \ \equiv \langle x, y\rangle \in \rho) \\
= \ & \langle \ (3.3) \ \rangle \\
& \rho = \tau \quad \Rightarrow \quad (\forall x, y \,| \ : true) \\
= \ & \langle \ (9.8) \text{ for multiple quantification } \rangle \\
& \rho = \tau \quad \Rightarrow \quad true \\
= \ & \langle \ (3.72) \ \rangle \\
& true \ .
\end{aligned}$$

Sufficiency follows using the Deduction Theorem.
Assume

$$(\forall x, y| : \langle x, y\rangle \in \rho \ \equiv \langle x, y\rangle \in \tau) \ .$$

Let z be a fresh variable. We prove

$$z \in \rho \quad \equiv \quad z \in \tau$$

from which

$$\rho \ = \ \tau \ .$$

$$\begin{aligned}
& z \in \rho \\
= \ & \langle \ 14 \text{ R} \ \rangle \\
& (z \in \rho) \ \wedge \ (\exists x, y \,| \ : z = \langle x, y\rangle\,) \\
= \ & \langle \ (9.21) \text{ for multiple quantification } \rangle \\
& (\exists x, y \,| \ : (z \in \rho) \ \wedge \ z = \langle x, y\rangle\,) \\
= \ & \langle \ (3.84)(a) \ \rangle \\
& (\exists x, y \,| \ : \langle x, y\rangle \in \rho \ \wedge \ z = \langle x, y\rangle\,) \\
= \ & \langle \ \text{Assumption} \ \rangle \\
& (\exists x, y \,| \ : \langle x, y\rangle \in \tau \ \wedge \ z = \langle x, y\rangle\,) \\
= \ & \langle \ (3.84)(a) \ \rangle \\
& (\exists x, y \,| \ : (z \in \tau) \ \wedge \ z = \langle x, y\rangle\,) \\
= \ & \langle \ (9.21) \text{ for multiple quantification } \rangle \\
& (z \in \tau) \ \wedge \ (\exists x, y \,| \ : z = \langle x, y\rangle\,) \\
= \ & \langle \ 14 \text{ R} \ \rangle \\
& z \in \tau \ .
\end{aligned}$$