Three Variations on a Theme of Dow

In his *Notes for January 17* Dow suggests the following exercise:

$$S \subset T \Rightarrow T \neq \emptyset \;.$$

Semantic Interlude: If $S \subset T$, T must contain an element which is not in S, so in particular, T contains an element.

We give a proof reflecting Dow's approach followed by three variations.

Dow's proof:

$$
\begin{array}{ll}
 & S \subset T \\
= & \quad \langle\ (11.14)\ \rangle \\
 & S \subseteq T \wedge S \neq T \\
= & \quad \langle\ (11.13),\ (11.4)\ \ \rangle \\
 & (\forall x|\, x \in S : x \in T) \wedge \neg(\forall x| : x \in S \equiv x \in T) \\
= & \quad \langle\ (9.18)(c)\ \rangle \\
 & (\forall x|\, x \in S : x \in T) \wedge (\exists x| : \neg(x \in S \equiv x \in T)) \\
= & \quad \langle\ (9.21)\ \rangle \\
 & (\exists x| : (\forall x| : x \in S \Rightarrow x \in T) \wedge \neg(x \in S \equiv x \in T)) \\
= & \quad \langle\ (3.80),\ (3{,}47)(b)\ \rangle \\
 & (\exists x| : (\forall x| : x \in S \Rightarrow x \in T) \wedge (\neg(x \in S \Rightarrow x \in T) \vee \neg(x \in T \Rightarrow x \in S))) \\
= & \quad \langle\ (3.46)\ \rangle \\
 & (\exists x| : ((\forall x| x \in S \Rightarrow x \in T) \wedge \neg(x \in S \Rightarrow x \in T)) \vee ((\forall x| : x \in S \Rightarrow x \in T) \wedge \neg(x \in T \Rightarrow x \in S))) \\
= & \quad \langle\ \text{Lemma: } (\forall x| : P) \wedge \neg Px \equiv \mathit{false}\ \rangle \\
 & (\exists x| : \mathit{false} \vee ((\forall x| : x \in S \Rightarrow x \in T) \wedge \neg(x \in T \Rightarrow x \in S))) \\
= & \quad \langle\ (3.30)\ \rangle \\
 & (\exists x| : (\forall x| : x \in S \Rightarrow x \in T) \wedge \neg(x \in T \Rightarrow x \in S)) \\
= & \quad \langle\ (9.21)\ \rangle \\
 & (\forall x| : x \in S \Rightarrow x \in T) \wedge (\exists x| : \neg(x \in T \Rightarrow x \in S)) \\
\Rightarrow & \quad \langle\ (3.76)(b)\ \rangle \\
 & (\exists x| : \neg(x \in T \Rightarrow x \in S)) \\
= & \quad \langle\ \text{Lemma: } \neg(p \Rightarrow q) \equiv p \wedge \neg q\ \rangle \\
 & (\exists x| : x \in T \wedge x \notin S)) \\
\Rightarrow & \quad \langle\ (9.26)\ x \in T \equiv (x \in T \wedge x \notin S) \vee x \in T \text{ by } (3.43)(b)\ \rangle \\
 & (\exists x| : x \in T)) \\
= & \quad \langle\ \ \text{Lemma: } (\exists x| : x \in T) \equiv T \neq \emptyset\ \ \rangle \\
 & T \neq \emptyset
\end{array}
$$

Variation 1: Using (8.15) instead of (9.21)

$$
\begin{array}{ll}
& S \subset T \\
= & \quad \langle\ (11.14)\ \rangle \\
& S \subseteq T \wedge S \neq T \\
= & \quad \langle\ (11.13),\ (11.4)\ \rangle \\
& (\forall x | \, x \in S : x \in T) \wedge \neg(\forall x | : x \in S \equiv x \in T) \\
= & \quad \langle\ (3.80)\ \rangle \\
& (\forall x | \, x \in S : x \in T) \wedge \neg(\forall x | : (x \in S \Rightarrow x \in T) \wedge (x \in T \Rightarrow x \in S)) \\
= & \quad \langle\ (8.15)\ \rangle \\
& (\forall x | \, x \in S : x \in T) \wedge \neg((\forall x | : (x \in S \Rightarrow x \in T) \wedge (\forall x | : x \in T \Rightarrow x \in S)) \\
= & \quad \langle\ (3.47)(a)\ \rangle \\
& (\forall x | \, x \in S : x \in T) \wedge (\neg(\forall x | : x \in S \Rightarrow x \in T) \vee \neg(\forall x | : x \in T \Rightarrow x \in S)) \\
= & \quad \langle\ (3.46)\ \rangle \\
& ((\forall x | \, x \in S : x \in T) \wedge \neg(\forall x | : x \in S \Rightarrow x \in T)) \vee ((\forall x | \, x \in S : x \in T) \wedge \neg(\forall x | : x \in T \Rightarrow x \in S)) \\
= & \quad \langle\ (9.4),\ (3.42)\ \rangle \\
& false \vee ((\forall x | \, x \in S : x \in T) \wedge \neg(\forall x | : x \in T \Rightarrow x \in S)) \\
= & \quad \langle\ (3.30)\ \rangle \\
& (\forall x | \, x \in S : x \in T) \wedge \neg(\forall x | : x \in T \Rightarrow x \in S) \\
\Rightarrow & \quad \langle\ (3.76)(b)\ \rangle \\
& \neg(\forall x | : x \in T \Rightarrow x \in S) \\
= & \quad \langle\ (9.4)\ \rangle \\
& \neg(\forall x | : x \in T : x \in S) \\
= & \quad \langle\ (9.18)(c)\ \rangle \\
& (\exists x | : x \in T : x \notin S) \\
= & \quad \langle\ (9.4) \text{ twice}\ \rangle \\
& (\exists x | x \notin S : x \in T) \\
\Rightarrow & \quad \langle\ (9.25) \text{ with } Q \text{ being } \mathit{true},\ (3.29)\ \rangle \\
& (\exists x | : x \in T) \\
= & \quad \langle\ \text{Lemma: } (\exists x | : x \in T) \equiv T \neq \emptyset\ \rangle \\
& T \neq \emptyset
\end{array}
$$

Variation 2: Proof by Contrapositive with use of Deduction Theorem

We prove

$$T = \emptyset \Rightarrow \neg(S \subset T) \ .$$

Assume $\vdash T = \emptyset$ from which one infers $\vdash x \in T \equiv \mathit{false}$ and $\vdash x \notin T \equiv \mathit{true}$.

$$\begin{array}{ll}
 & \neg(S \subset T) \\
= & \quad \langle\ (11.14)\ \rangle \\
 & \neg(S \subseteq T \wedge S \neq T) \\
= & \quad \langle\ (11.13)\ \rangle \\
 & \neg((\forall x |\, x \in S : x \in T) \wedge S \neq T) \\
= & \quad \langle\ (9.18)(c)\ \rangle \\
 & (\exists x |\, x \in S : x \notin T) \vee S = T \\
= & \quad \langle\ (9.19) \text{ twice, } (11.4)\ \rangle \\
 & (\exists x |\, x \notin T : x \in S) \vee (\forall x | : x \in S \equiv x \in T) \\
= & \quad \langle\ \text{Assumption } x \in T \equiv \mathit{false}\ \rangle \\
 & (\exists x |\, \mathit{true} : x \in S) \vee (\forall x | : x \in S \equiv \mathit{false}) \\
= & \quad \langle\ (3.15)\ \rangle \\
 & (\exists x | : x \in S) \vee (\forall x | : x \notin S) \\
= & \quad \langle\ (9.18)(b)\ \rangle \\
 & (\exists x | : x \in S) \vee \neg(\exists x | : x \in S)
\end{array}$$

which is (3.28).

Variation 3: Proof using Metatheorem Witness

As in Dow's proof

$$
\begin{array}{ll}
 & S \subset T \Rightarrow T \neq \emptyset \\
= & \quad \langle\ (11.14)\ \rangle \\
 & S \subseteq T \wedge S \neq T \Rightarrow T \neq \emptyset \\
= & \quad \langle\ (11.13),\ (11.4)\ \ \rangle \\
 & (\forall x | \, x \in S : x \in T) \wedge \neg (\forall x | : x \in S \equiv x \in T) \Rightarrow T \neq \emptyset \\
= & \quad \langle\ (9.18)(c)\ \rangle \\
 & (\forall x | \, x \in S : x \in T) \wedge (\exists x | : \neg (x \in S \equiv x \in T)) \Rightarrow T \neq \emptyset \\
= & \quad \langle\ (9.21)\ \rangle \\
 & (\exists x | : (\forall x | : (x \in S \Rightarrow x \in T)) \wedge \neg (x \in S \equiv x \in T)) \Rightarrow T \neq \emptyset
\end{array}
$$

By Metatheorem Witness, the last line is a theorem, if and only if

$$\vdash (\forall x | : (x \in S \Rightarrow x \in T)) \wedge \neg (w \in S \equiv w \in T) \Rightarrow T \neq \emptyset \ .$$

$$
\begin{array}{ll}
 & (\forall x | : (x \in S \Rightarrow x \in T)) \wedge \neg (w \in S \equiv w \in T) \\
\Rightarrow & \quad \langle\ (9.13), \text{Lemma: If} \vdash P \Rightarrow Q \text{ then} \vdash P \wedge R \Rightarrow Q \wedge R\ \rangle \\
 & (w \in S \Rightarrow w \in T) \wedge \neg (w \in S \equiv w \in T) \\
\Rightarrow & \quad \langle\ (3.9)\ \rangle \\
 & (w \in S \Rightarrow w \in T) \wedge (w \notin S \equiv w \in T) \\
= & \quad \langle\ (3.80)\ \rangle \\
 & (w \in S \Rightarrow w \in T) \wedge (w \notin S \Rightarrow w \in T) \wedge (w \in T \Rightarrow w \notin S) \\
\Rightarrow & \quad \langle\ (3.76)(b)\ \rangle \\
 & (w \in S \Rightarrow a \in T) \wedge (w \notin S \Rightarrow w \in T) \\
= & \quad \langle\ (3.79)\ \rangle \\
 & a \in T \\
\Rightarrow & \quad \langle\ (9.28)\ \rangle \\
 & (\exists x | : x \in T) \\
= & \quad \langle\ \ \text{Lemma: } (\exists x | : x \in T) \equiv T \neq \emptyset\ \ \rangle \\
 & T \neq \emptyset
\end{array}
$$