York University

Faculty of Arts, Faculty of Science. Atkinson College

Math 2090

Class Test 2

SOLUTIONS

Instructions:

1. Time allowed : 75 minutes.
2. There are 4 questions on 5 pages.
3. Answer all questions.
4. Prove, means prove using the proof methods and proof format of the text.
   If your reason is not a theorem on the list provided, or another question on this test, you must provide its proof.
5. Your reasons must indicate that you have verified any conditions concerning free occurrences of variables.

Question	**Points**	**Marks**
1	8	
2	8	
3	10	
4	14	
Total	40	

1. (8 points) Prove that,

$$\vdash\ S \cap T \neq \emptyset \ \Rightarrow\ (S \times T) \cap (T \times S)\ \neq\ \emptyset\ .$$

Answer: Take z, w to be fresh variables.

$$\begin{array}{ll} & S \cap T \neq \emptyset \ \Rightarrow\ (S \times T) \cap (T \times S)\ \neq\ \emptyset \\ = & \langle\ \text{Property of } \emptyset\ \rangle \\ & (\exists z \mid\ : z \in S \cap T)\ \Rightarrow\ (S \times T) \cap ((T \times S))\ \neq\ \emptyset\ . \end{array}$$

By (9.30) it suffices to prove that, $\vdash\ w \in S \cap T\ \Rightarrow\ (S \times T) \cap ((T \times S))\ \neq\ \emptyset$.
Assume $w \in S \cap T$.

$$\begin{array}{ll} & (S \times T) \cap (T \times S)\ \neq\ \emptyset \\ = & \langle\ \text{Property of } \emptyset\ \rangle \\ & (\exists z \mid\ : z \in (S \times T) \cap (T \times S)) \\ = & \langle\ (9.17),\ (3.15)\ \rangle \\ & \neg(\forall z \mid\ : z \in (S \times T) \cap (T \times S) \equiv false) \\ = & \langle\ (9.13),\ (3.60)\ \rangle \\ & \neg((\forall z \mid\ : z \in (S \times T) \cap (T \times S) \equiv false)\ \wedge\ (\langle w, w\rangle \in (S \times T) \cap (T \times S) \equiv false)) \\ = & \langle\ (11.21)\ \rangle \\ & \neg((\forall z \mid\ : z \in (S \times T) \cap (T \times S) \equiv false)\ \wedge\ (\langle w, w\rangle \in (S \times T) \wedge \langle w, w\rangle \in (T \times S) \equiv false) \\ = & \langle\ (14.4)\ \rangle \\ & \neg((\forall z \mid\ : z \in (S \times T) \cap (T \times S) \equiv false)\ \wedge\ (w \in S \wedge w \in T \wedge w \in T \wedge w \in S \equiv false)) \\ = & \langle\ (3.38)\ \rangle \\ & \neg((\forall z \mid\ : z \in (S \times T) \cap (T \times S) \equiv false)\ \wedge\ (w \in S \wedge w \in T \equiv false)) \\ = & \langle\ \text{Temporary theorem, } w \in S \wedge w \in T\ \rangle \\ & \neg((\forall z \mid\ : z \in (S \times T) \cap (T \times S) \equiv false)\ \wedge\ (true \equiv false)) \\ = & \langle\ (3.3)\ \rangle \\ & \neg((\forall z \mid\ : z \in (S \times T) \cap (T \times S) \equiv false)\ \wedge\ false) \\ = & \langle\ (3.47)(\text{a})\ \rangle \\ & \neg(\forall z \mid\ : z \in (S \times T) \cap (T \times S) \equiv false) \vee true \\ = & \langle\ (3.29)\ \rangle \\ & true\ . \end{array}$$

2. (8 points) Prove that for ρ a relation and σ a relation,

$$\vdash \ Ran.(\rho \circ \sigma) \subseteq Ran.\sigma \ .$$

Answer: Take x, y, z to be fresh variables.

$$\begin{array}{cl} & Ran.(\rho \circ \sigma) \subseteq Ran.\sigma \\ = & \langle \ (11.3) \ \rangle \\ & (\forall y \,|\, y \in Ran.(\rho \circ \sigma) : y \in Ran.\sigma) \\ = & \langle \ (9.2) \ \rangle \\ & (\forall y \,|\, : y \in Ran.(\rho \circ \sigma) \Rightarrow y \in Ran.\sigma) \ . \end{array}$$

By (9.16) it suffices to prove that, $\vdash \ y \in Ran.(\rho \circ \sigma) \Rightarrow y \in Ran.\sigma$.

$$\begin{array}{cl} & y \in Ran.(\rho \circ \sigma) \\ = & \langle \ (14.17) \ \rangle \\ & (\exists x \,|\, : \langle x, y \rangle \in \rho \circ \sigma) \\ = & \langle \ (14.20),\ z \text{ d.n.o.f. } \rho, \sigma \ \rangle \\ & (\exists x \,|\, : (\exists z \,|\, : \langle x, z \rangle \in \rho \wedge \langle z, y \rangle \in \sigma)) \\ = & \langle \ (8.19),\ x \text{ d.n.o.f. } true,\ z \text{ d.n.o.f. } true \ \rangle \\ & (\exists z \,|\, : (\exists x \,|\, : \langle x, z \rangle \in \rho \wedge \langle z, y \rangle \in \sigma)) \\ = & \langle \ (9.21) \ \rangle \\ & (\exists z \,|\, : (\exists x \,|\, : \langle x, z \rangle \in \rho) \wedge \langle z, y \rangle \in \sigma) \\ \Rightarrow & \langle \ (9.26)' \ \rangle \\ & (\exists z \,|\, : \langle z, y \rangle \in \sigma) \\ = & \langle \ (14.17),\ z \text{ d.n.o.f. } \sigma \ \rangle \\ & y \in Ran.\sigma \ . \end{array}$$

3. (10 points) Use Mathematical Induction to prove that,

$$\vdash \rho^2 \subseteq \rho \;\Rightarrow\; (\forall n \,|\, n \geq 1 : \rho^n \subseteq \rho) \;.$$

Answer: Assume $\rho^2 \subseteq \rho$. We need to prove $(\forall n \,|\, n \geq 1 : \rho^n \subseteq \rho)$.
The base case is when $n = 1$, i.e., $\rho^1 \subseteq \rho$.

Take z to be a fresh variable.

$$\begin{aligned}
& \rho^1 \subseteq \rho \\
= \; & \langle \; (14.25) \; \rangle \\
& \rho \subseteq \rho \\
= \; & \langle \; (11.13) \; \rangle \\
& (\forall z \,|\, z \in \rho : z \in \rho) \\
= \; & \langle \; (9.2) \; \rangle \\
& (\forall z \,|\; : z \in \rho \Rightarrow z \in \rho) \\
= \; & \langle \; (3.71) \; \rangle \\
& (\forall z \,|\; : \mathit{true}) \\
= \; & \langle \; (9.8) \; \rangle \\
& \mathit{true} \;.
\end{aligned}$$

Take x, y to be fresh variables. Assume $n \geq 1 \;\Rightarrow\; \rho^n \subseteq \rho$. We need to prove $n \geq 1 \;\Rightarrow\; \rho^{n+1} \subseteq \rho$. Assume $n \geq 1$.

$$\begin{aligned}
& \rho^{n+1} \subseteq \rho \\
= \; & \langle \; (14.25) \; \rangle \\
& \rho^n \circ \rho \subseteq \rho \\
= \; & \langle \; (11.13) \; \rangle \\
& (\forall z \,|\, z \in \rho^n \circ \rho : z \in \rho) \\
= \; & \langle \; \text{14 RP, (9.2) for multiple quantifications} \; \rangle \\
& (\forall x, y \,|\; : \langle x, y\rangle \in \rho^n \circ \rho \;\Rightarrow\; \langle x, y\rangle \in \rho) \\
= \; & \langle \; (14.20) \; \rangle \\
& (\forall x, y \,|\; : (\exists z \,|\, \langle x, z\rangle \in \rho^n \wedge \langle z, y\rangle \in \rho) \;\Rightarrow\; \langle x, y\rangle \in \rho) \;.
\end{aligned}$$

Observe that x, y do not occur free in the Assumption.
By (9.16) it suffices to prove $(\exists z \,|\; : \langle x, z\rangle \in \rho^n \wedge \langle z, y\rangle \in \rho) \;\Rightarrow\; \langle x, y\rangle \in \rho$.

$$\begin{aligned}
& (\exists z \,|\; : \langle x, z\rangle \in \rho^n \wedge \langle z, y\rangle \in \rho)) \\
\Rightarrow \; & \langle \; \text{Temporary theorem, } \langle x, z\rangle \in \rho^n \Rightarrow \langle x, z\rangle \in \rho\text{, (9.27), (9.16), Modus Ponens} \; \rangle \\
& (\exists z \,|\; : \langle x, z\rangle \in \rho \wedge \langle z, y\rangle \in \rho)) \\
= \; & \langle \; (14.20), (14.25) \; \rangle \\
& \langle x, y\rangle \in \rho^2 \\
\Rightarrow \; & \langle \; \text{Temporary theorem, } \langle x, y\rangle \in \rho^2 \Rightarrow \langle x, y\rangle \in \rho \; \rangle \\
& \langle x, y\rangle \in \rho \;.
\end{aligned}$$

4. (a) (6 points) Prove that,

$$\vdash \rho \subseteq X \times Y \;\Rightarrow\; (\langle x, y\rangle \in \rho \;\Rightarrow\; x \in X)\ .$$

Answer: Take z, u, v to be fresh variables.

$$
\begin{array}{ll}
 & \rho \subseteq X \times Y \\
= & \langle\ (11.13)\ \rangle \\
 & (\forall z \mid z \in \rho : z \in X \times Y) \\
= & \langle\ 14\ \text{RP}\ \rangle \\
 & (\forall u, v \mid \langle u, v\rangle \in \rho : \langle u, v\rangle \in X \times Y) \\
= & \langle\ (14.4)\ \rangle \\
 & (\forall u, v \mid \langle u, v\rangle \in \rho : u \in X \wedge v \in Y) \\
= & \langle\ (9.2)\ \text{for multiple quantifications}\ \rangle \\
 & (\forall u, v \mid\ : \langle u, v\rangle \in \rho \Rightarrow u \in X \wedge v \in Y) \\
= & \langle\ (8.20),\ v\ \text{d.n.o.f.}\ \mathit{true}\ \rangle \\
 & (\forall u \mid\ : (\forall v \mid\ : \langle u, v\rangle \in \rho \Rightarrow u \in X \wedge v \in Y)) \\
\Rightarrow & \langle\ (9.13),\ v\ \text{d.n.o.f.}\ x\ \rangle \\
 & (\forall v \mid\ : \langle x, v\rangle \in \rho \Rightarrow x \in X \wedge v \in Y) \\
\Rightarrow & \langle\ (9.13)\ \rangle \\
 & \langle x, y\rangle \in \rho \Rightarrow x \in X \wedge y \in Y \\
\Rightarrow & \langle\ \vdash (P \Rightarrow Q \wedge R) \;\Rightarrow\; (P \Rightarrow Q)\ \text{via (3.59), (3.76)(d)}\ \rangle \\
 & \langle x, y\rangle \in \rho \Rightarrow x \in X\ .
\end{array}
$$

(b) (8 points) Prove that,

$$\vdash\ \rho \subseteq X \times Y\ \wedge\ \imath = \{w \mid w \in X : \langle w, w\rangle\}\quad \Rightarrow\quad \imath \circ \rho = \rho\ .$$

Answer: Take x, y, z to be fresh variables. Assume $\rho \subseteq X \times Y\ \wedge\ \imath = \{w \mid w \in X : \langle w, w\rangle\}$.

$$\begin{aligned}
& \imath \circ \rho = \rho \\
= \ & \langle\ \text{14 RE}\ \rangle \\
& (\forall x, y \mid \langle x, y\rangle \in \imath \circ \rho\ \equiv\ \langle x, y\rangle \in \rho)\ .
\end{aligned}$$

By (9.16) for multiple quantifications, it suffices to prove that, $\vdash\ \langle x, y\rangle \in \imath \circ \rho\ \equiv\ \langle x, y\rangle \in \rho$.

$$\begin{aligned}
& \langle x, y\rangle \in \imath \circ \rho \\
= \ & \langle\ \text{(14.20)},\ z \text{ d.n.o.f. } \imath, \rho\ \rangle \\
& (\exists z \mid\ : \langle x, z\rangle \in \imath \wedge \langle z, y\rangle \in \rho) \\
= \ & \langle\ \text{Assumption, (11.3)}\ \rangle \\
& (\exists z \mid\ : (\exists w \mid w \in X : \langle x, z\rangle = \langle w, w\rangle) \wedge \langle z, y\rangle \in \rho) \\
= \ & \langle\ \text{(14.4)}\ \rangle \\
& (\exists z \mid\ : (\exists w \mid w \in X : x = w \wedge z = w) \wedge \langle z, y\rangle \in \rho) \\
= \ & \langle\ \text{(9.19) twice}\ \rangle \\
& (\exists z \mid\ : (\exists w \mid w = z : w \in X \wedge x = w) \wedge \langle z, y\rangle \in \rho) \\
= \ & \langle\ \text{(8.14)},\ w \text{ d.n.o.f. } z\ \rangle \\
& (\exists z \mid\ : z \in X \wedge x = z \wedge \langle z, y\rangle \in \rho) \\
= \ & \langle\ \text{(9.19)}\ \rangle \\
& (\exists z \mid z = x : z \in X \wedge \langle z, y\rangle \in \rho) \\
= \ & \langle\ \text{(8.14)},\ z \text{ d.n.o.f. } x\ \rangle \\
& x \in X \wedge \langle x, y\rangle \in \rho \\
= \ & \left\langle \begin{array}{l} \text{Temporary theorem, } \rho \subseteq X \times Y \text{, gives by (a),} \\ \langle x, y\rangle \in \rho \Rightarrow x \in X \text{, (3.60)} \end{array} \right\rangle \\
& \langle x, y\rangle \in \rho\ .
\end{aligned}$$

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