

Math 2090 A  
 Quiz 2 Version 2    October 31, 2002  
 SOLUTIONS

1. (10 points) Define sets  $T_n$  inductively by

$$\begin{aligned} T_0 &= \emptyset \\ T_{n+1} &= T_n \cup \{x_{n+1}\} \text{ for } n \geq 0. \end{aligned}$$

Use Mathematical Induction and give a detailed proof that

$$\vdash (\forall n \mid n \geq 0 : \#T_n \leq n).$$

**Answer:** Take  $x$  to be a fresh variable.

For the base case

$$\begin{aligned} &\#T_0 \leq 0 \\ &= \langle \text{Definition of } T_0, (3.83), \text{ Modus Ponens} \rangle \\ &\#\emptyset \leq 0 \\ &= \langle (11.12) \rangle \\ &(+x \mid x \in \emptyset : 1) \leq 0 \\ &= \langle \vdash x \in \emptyset \equiv \text{false} \rangle \\ &(+x \mid \text{false} : 1) \leq 0 \\ &= \langle (8.13) \rangle \\ &0 \leq 0. \end{aligned}$$

Assume  $\#T_n \leq n$ .

$$\begin{aligned} &\#T_{n+1} \\ &= \langle \text{Inductive Definition, (3.83), Modus Ponens} \rangle \\ &\quad \#(T_n \cup \{x_{n+1}\}) \\ &= \langle (11.12) \rangle \\ &\quad (+x \mid x \in T_n \cup \{x_{n+1}\} : 1) \\ &= \langle (11.20) \rangle \\ &\quad (+x \mid x \in T_n \vee x \in \{x_{n+1}\} : 1) \\ &= \langle (11.17), \text{ Arithmetic} \rangle \\ &\quad (+x \mid x \in T_n : 1) + (+x \mid x \in \{x_{n+1}\} : 1) - (+x \mid x \in T_n \wedge x \in \{x_{n+1}\} : 1) \\ &\leq \langle \text{Arithmetic, } \vdash (+x \mid x \in T_n \wedge x \in \{x_{n+1}\} : 1) \geq 0 \rangle \\ &\quad (+x \mid x \in T_n : 1) + (+x \mid x \in \{x_{n+1}\} : 1) \\ &= \langle \text{Useful Lemma} \rangle \\ &\quad (+x \mid x \in T_n : 1) + (+x \mid x = x_{n+1} : 1) \\ &= \langle (8.14) \rangle \end{aligned}$$

$$\begin{array}{c} \#T_n + 1 \\ \leq \langle \text{Assumption, Arithmetic} \rangle \\ n + 1 \end{array}$$

2. (2 points) Clearly state the Mathematical Induction Inference Rule you used in Part (a).

**Answer:**

$$\frac{\vdash P.0, \vdash P.n \Rightarrow P.n + 1}{\vdash (\forall n | : P.n)} .$$

3. (3 points) Give a careful syntactic proof that  $\vdash \#T_{314} \leq 314$ .

**Answer:** As  $n$  is of type  $\mathbb{N}$ , we have  $n \geq 0 \equiv \text{true}$ .

By (9.13),  $\vdash (\forall n | : \#T_n \leq n) \Rightarrow \#T_{314} \leq 314$ .

By Part (a) and Modus Ponens,  $\vdash \#T_{314} \leq 314$ .