

York University

Faculty of Arts, Faculty of Science. Atkinson College

Math 2090

Class Test 1

SOLUTIONS

Instructions:

1. Time allowed : 75 minutes.
2. There are 4 questions on 6 pages. Page 2 is blank.
3. Answer all questions.
4. Prove, means prove using the proof methods and proof format of the text.
If your reason is not a theorem on the list provided, or another question on this test, you must provide its proof.
5. Your reasons must indicate that you have verified any conditions concerning free occurrences of variables.

Question	Points	Marks
1	9	
2	9	
3	13	
4	9	
Total	40	

1. (a) (3 points) Prove that,

$$\vdash w \in \{x, y\} \equiv w = x \vee w = y .$$

Answer:

$$\begin{aligned} & w \in \{x, y\} \\ = & \langle \text{Abbreviation} \rangle \\ & w \in \{w \mid w = x \vee w = y\} \\ = & \langle (11.7) \rangle \\ & w = x \vee w = y . \end{aligned}$$

- (b) (6 points) Prove that,

$$\vdash \{x, y\} = \{y, z\} \equiv x = z .$$

Answer: Take w to be a fresh variable.

$$\begin{aligned} & \{x, y\} = \{y, z\} \\ = & \langle (11.4), w \text{ d.n.o.f. } \{x, y\}, \{y, z\} \rangle \\ & (\forall w \mid : w \in \{x, y\} \equiv w \in \{y, z\}) \\ = & \langle \text{Part (a)} \rangle \\ & (\forall w \mid : w = x \vee w = y \equiv w = y \vee w = z) \\ = & \langle (9.13), (3.60) \rangle \\ & (\forall w \mid : w = x \vee w = y \equiv w = y \vee w = z) \\ & \wedge (x = x \vee x = y \equiv x = y \vee x = z) \wedge (z = x \vee z = y \equiv z = y \vee z = z) \\ = & \langle (1.2), (3.29), (3.3) \rangle \\ & (\forall w \mid : w = x \vee w = y \equiv w = y \vee w = z) \wedge (x = y \vee x = z) \wedge (z = x \vee z = y) \\ = & \langle (3.45) \rangle \\ & (\forall w \mid : w = x \vee w = y \equiv w = y \vee w = z) \wedge (x = z \vee (x = y \wedge z = y)) \\ = & \langle (1.4), (3.60) \rangle \\ & (\forall w \mid : w = x \vee w = y \equiv w = y \vee w = z) \wedge (x = z \vee (x = y \wedge z = y \wedge x = z)) \\ = & \langle (3.43)(b) \rangle \\ & (\forall w \mid : w = x \vee w = y \equiv w = y \vee w = z) \wedge x = z \\ = & \langle (3.84)(a) \rangle \end{aligned}$$

$$\begin{aligned} & (\forall w \mid : w = z \vee w = y \equiv w = y \vee w = z) \wedge x = z \\ = & \langle (3.3) \rangle \\ & (\forall w \mid : \text{true}) \wedge x = z \\ = & \langle (9.8) \rangle \\ & \text{true} \wedge x = z \\ = & \langle (3.39) \rangle \\ & x = z . \end{aligned}$$

2. (a) (6 points) Prove that, provided x does not occur free in S, T ,

$$\vdash (\exists x | x \in S : x \notin T) \Rightarrow S \neq T .$$

Answer: By Contrapositive, it suffices to prove,

$$S = T \Rightarrow \neg(\exists x | x \in S : x \notin T) .$$

$$\begin{aligned} & S = T \Rightarrow \neg(\exists x | x \in S : x \notin T) \\ = & \langle (9.18)(a) \rangle \\ & S = T \Rightarrow (\forall x | x \in S : x \in T) \\ = & \langle (9.2) \rangle \\ & S = T \Rightarrow (\forall x | : x \in S \Rightarrow x \in T) . \end{aligned}$$

Now,

$$\begin{aligned} & S = T \\ = & \langle (11.4), x \text{ d.n.o.f. } S, T \rangle \\ & (\forall x | : x \in S \equiv x \in T) \\ = & \langle (3.80) \rangle \\ & (\forall x | : (x \in S \Rightarrow x \in T) \wedge (x \in T \Rightarrow x \in S)) \\ = & \langle (8.15) \rangle \\ & (\forall x | : x \in S \Rightarrow x \in T) \wedge (\forall x | : x \in T \Rightarrow x \in S) \\ \Rightarrow & \langle (3.76)(b) \rangle \\ & (\forall x | : x \in S \Rightarrow x \in T) . \end{aligned}$$

- (b) (3 points) For x not occurring free in S, T , is

$$(\exists x | x \in S : x \notin T) \equiv S \neq T$$

a theorem? Explain carefully.

Answer: It is not a theorem. The following gives a state in which it is *f*. Take $S = \emptyset$ and $T = \{y\}$ with y not x . Then $(\exists x | x \in S : x \notin T)$ is *f* and $S \neq T$ is *t*.

3. (a) (3 points) Prove the “Useful Lemma”,

$$\vdash x \in \{y\} \equiv x = y .$$

Answer:

$$\begin{aligned} & x \in \{y\} \\ = & \langle \text{Abbreviation} \rangle \\ & x \in \{x \mid x = y\} \\ = & \langle (11.7) \rangle \\ & x = y \end{aligned}$$

- (b) (5 points) Prove that,

$$\vdash w \in S \Rightarrow \#(S \cup \{w\}) = \#S .$$

Answer: Assume $w \in S$. Take x to be a fresh variable.

$$\begin{aligned} & \#(S \cup \{w\}) \\ = & \langle (11.12), x \text{ d.n.o.f. } S \cup \{w\} \rangle \\ & (+x \mid x \in S \cup \{w\} : 1) \\ = & \langle \text{Lemma: } \vdash w \in S \Rightarrow S = S \cup \{w\}, \text{ Modus Ponens} \rangle \\ & (+x \mid x \in S : 1) \\ = & \langle (11.12), x \text{ d.n.o.f. } S \rangle \\ & \#S \end{aligned}$$

Now, to prove $\vdash w \in S \Rightarrow S = S \cup \{w\}$.

Assume $w \in S$. Take x to be a fresh variable.

$$\begin{aligned} & S = S \cup \{w\} \\ = & \langle (11.4), x \text{ d.n.o.f. } S, S \cup \{w\} \rangle \\ & (\forall x \mid : x \in S \equiv x \in S \cup \{w\}) \\ = & \langle (11.20) \rangle \\ & (\forall x \mid : x \in S \equiv x \in S \vee x \in \{w\}) \\ = & \langle \text{Useful Lemma} \rangle \\ & (\forall x \mid : x \in S \equiv x \in S \vee x = w) \\ = & \langle (11.57) \rangle \\ & (\forall x \mid : x = w \Rightarrow x \in S) \\ = & \langle (3.84)(b) \rangle \end{aligned}$$

$$\begin{aligned}
& (\forall x \mid : x = w \Rightarrow w \in S) \\
= & \langle \text{Assumption} \rangle \\
& (\forall x \mid : x = w \Rightarrow \text{true}) \\
= & \langle (3.72) \rangle \\
& (\forall x \mid : \text{true}) \\
= & \langle (9.8) \rangle \\
& \text{true}
\end{aligned}$$

(c) (5 points) Prove that,

$$\vdash w \notin S \Rightarrow \#(S \cup \{w\}) = \#S + 1 .$$

Answer: Assume $w \notin S$. Take x to be a fresh variable.

$$\begin{aligned}
& \#(S \cup \{w\}) \\
= & \langle (11.12), x \text{ d.n.o.f. } S \cup \{w\} \rangle \\
& (+x \mid x \in S \cup \{w\} : 1) \\
= & \langle (11.20) \rangle \\
& (+x \mid x \in S \vee x \in \{w\} : 1) \\
= & \left\langle \begin{array}{l} \text{Lemma, } \vdash w \notin S \Rightarrow (x \in S \wedge x \in \{w\} \equiv \text{false}), \\ \text{Modus Ponens, Strong (9.16), (8.16), Modus Ponens} \end{array} \right\rangle \\
& (+x \mid x \in S : 1) + (+x \mid x \in \{w\} : 1) \\
= & \langle (11.12), x \text{ d.n.o.f. } S, \text{ Useful Lemma} \rangle \\
& \#S + (+x \mid x = w : 1) \\
= & \langle (8.14) \rangle \\
& \#S + 1 .
\end{aligned}$$

Now, to prove $\vdash w \notin S \Rightarrow (x \in S \wedge x \in \{w\} \equiv \text{false})$.
Assume $w \notin S$.

$$\begin{aligned}
& x \in S \wedge x \in \{w\} \\
= & \langle \text{Useful Lemma} \rangle \\
& x \in S \wedge x = w \\
= & \langle (3.84)(a) \rangle \\
& w \in S \wedge x = w \\
= & \langle \text{Assumption} \rangle \\
& \text{false} \wedge x = w \\
= & \langle (3.40) \rangle \\
& \text{false} .
\end{aligned}$$

4. (a) (3 points) Prove that,

$$\vdash x \in S \equiv \{x\} \subseteq S .$$

Answer: Take w to be a fresh variable.

$$\begin{aligned} & \{x\} \subseteq S \\ = & \langle (11.13), w \text{ d.n.o.f. } \{x\}, S \rangle \\ & (\forall w \mid w \in \{x\} : w \in S) \\ = & \langle \text{Useful Lemma} \rangle \\ & (\forall w \mid w = x : w \in S) \\ = & \langle (8.14) \rangle \\ & x \in S . \end{aligned}$$

(b) (6 points) Prove that,

$$\vdash \mathcal{P}S \subseteq \mathcal{P}T \Rightarrow S \subseteq T .$$

Answer: Assume $\mathcal{P}S \subseteq \mathcal{P}T$. Take x to be a fresh variable.

$$\begin{aligned} & S \subseteq T \\ = & \langle (11.13), x \text{ d.n.o.f. } S, T \rangle \\ & (\forall x \mid x \in S : x \in T) \\ = & \langle \text{Part (a)} \rangle \\ & (\forall x \mid \{x\} \subseteq S : \{x\} \subseteq T) \\ = & \langle (9.2) \rangle \\ & (\forall x \mid : \{x\} \subseteq S \Rightarrow \{x\} \subseteq T) \\ = & \langle \text{Assumption, (11.14), (11.23), (9.13), Modus Ponens} \rangle \\ & (\forall x \mid : \text{true}) \\ = & \langle (9.8) \rangle \\ & \text{true} . \end{aligned}$$

The end