

# 1090 Review Exercise List SOLUTIONS

1. Give three different proofs of

$$\vdash (By \Rightarrow Ay) \Rightarrow (By \Rightarrow (\exists x | : Ax)) .$$

(a) Use a standard stacked (textbook style) proof with = on the left.

**Answer:**

$$\begin{aligned} & (By \Rightarrow Ay) \Rightarrow (By \Rightarrow (\exists x | : Ax)) \\ = & \langle (3.65) \rangle \\ & By \wedge (By \Rightarrow Ay) \Rightarrow (\exists x | : Ax) \\ = & \langle (3.66) \rangle \\ & By \wedge Ay \Rightarrow (\exists x | : Ax) \\ = & \langle (3.60) \rangle \\ & By \wedge Ay \wedge (\exists x | : Ax) \equiv By \wedge Ay \\ = & \langle (9.28), (3.60) \rangle \\ & By \wedge Ay \equiv By \wedge Ay . \end{aligned}$$

(b) Prove using Contrapositive. In proving the contrapositive, start with the antecedent and use a proof in the style of Chapter 4.1 with a mix of = and  $\Rightarrow$  symbols on the left.

**Answer:** The contrapositive is

$$\neg(By \Rightarrow (\exists x | : Ax)) \Rightarrow \neg(By \Rightarrow Ay) .$$

$$\begin{aligned} & \neg(By \Rightarrow (\exists x | : Ax)) \\ = & \langle (3.59) \rangle \\ & \neg(\neg By \vee (\exists x | : Ax)) \\ = & \langle (3.47)(b), (3.12) \rangle \\ & By \wedge \neg(\exists x | : Ax) \\ = & \langle (9.18)(b) \rangle \\ & By \wedge (\forall x | : \neg Ax) \\ \Rightarrow & \langle (9.13), (4.3), \text{Modus Ponens} \rangle \\ & By \wedge \neg Ay \\ = & \langle (3.47)(b), (3.12) \rangle \\ & \neg(\neg By \vee Ay) \\ = & \langle (3.59) \rangle \\ & \neg(By \Rightarrow Ay) . \end{aligned}$$

(c) Prove using the Deduction Theorem.

**Answer:**

Assume  $By \Rightarrow Ay$ . We need to prove  $By \Rightarrow (\exists x| : Ax)$ .

Assume  $By$ .

Since we have  $By \Rightarrow Ay$ , by Strong Modus Ponens, we have  $Ay$ .

But by (9.28),  $\vdash Ay \Rightarrow (\exists x| : Ax)$ .

By Modus Ponens,  $(\exists x| : Ax)$ .

2. Determine whether or not

$$(By \Rightarrow Ay) \Rightarrow ((\exists x| : Bx) \Rightarrow (\exists x| : Ax))$$

is a theorem. If yes, give a proof. If no, give an interpretation for which it is false.

**Answer:**

The expression is not a theorem. Take the universe of discourse to be  $\{0, 1\}$ . Take  $A$  to never hold,  $B$  to hold for 0 but not for 1. Assign 1 to the free occurrences of  $y$ .

$By$  is false.

$By \Rightarrow Ay$  is true.

$(\exists x| : Bx)$  is true.

$(\exists x| : Ax)$  is false.

$(\exists x| : Bx) \Rightarrow (\exists x| : Ax)$  is false.

$(By \Rightarrow Ay) \Rightarrow ((\exists x| : Bx) \Rightarrow (\exists x| : Ax))$  is false.

3. Prove this instance of (9.27),

$$\vdash (\forall y| : By \Rightarrow Ay) \Rightarrow ((\exists x| : Bx) \Rightarrow (\exists x| : Ax)) .$$

Start by using (3.65) to obtain a form to which (9.30) applies, apply (9.30) and complete a proof.

**Answer:** As indicated,

$$\begin{aligned} & (\forall x| : By \Rightarrow Ay) \Rightarrow ((\exists x| : Bx) \Rightarrow (\exists x| : Ax)) \\ = & \langle (3.65) \rangle \\ & (\forall x| : By \Rightarrow Ay) \wedge (\exists x| : Bx) \Rightarrow (\exists x| : Ax) \\ = & \langle (3.65) \rangle \\ & (\exists x| : Bx) \Rightarrow ((\forall y| : By \Rightarrow Ay) \Rightarrow (\exists x| : Ax)) , \end{aligned}$$

so it suffices to prove

$$\vdash (\exists x| : Bx) \Rightarrow ((\forall y| : By \Rightarrow Ay) \Rightarrow (\exists x| : Ax)) .$$

Take  $w$  to be a fresh variable. By (9.30) it suffices to prove

$$\vdash Bw \Rightarrow ((\forall y| : By \Rightarrow Ay) \Rightarrow (\exists x| : Ax)) ,$$

or equivalently by (3.65) and Equanimity,

$$\vdash Bw \wedge (\forall y| : By \Rightarrow Ay) \Rightarrow (\exists x| : Ax) .$$

$$\begin{aligned}
& Bw \wedge (\forall y | : By \Rightarrow Ay) \\
= & \langle (9.13), (3.60) \rangle \\
& Bw \wedge (\forall y | : By \Rightarrow Ay) \wedge (Bw \Rightarrow Aw) \\
= & \langle (3.66) \rangle \\
& Bw \wedge Aw \wedge (\forall y | : By \Rightarrow Ay) \\
\Rightarrow & \langle (3.76)(b) \rangle \\
& Aw \\
\Rightarrow & \langle (9.28) \rangle \\
& (\exists x | : Ax) .
\end{aligned}$$

4. (a) Prove  $\vdash (+j | 2 \leq j \leq n : -a[j - 1]) = (+j | 1 \leq j \leq n - 1 : -a[j])$ .  
Note: Do not use (8.22). Review the proof steps for (8.22) given in the text and incorporate them in your proof.

**Answer:** Choose  $k$  to be fresh.

$$\begin{aligned}
& (+j | 2 \leq j \leq n : -a[j - 1]) \\
= & \langle (8.14), k \text{ d.n.o.f. in } j - 1 \rangle \\
& (+j | 2 \leq j \leq n : (+k | k = j - 1 : -a[k])) \\
= & \langle (8.20), k \text{ d.n.o.f. in } 2 \leq j \leq n \rangle \\
& (+j, k | (2 \leq j \leq n) \wedge (k = j - 1) : -a[k]) \\
= & \langle \text{Arithmetic} \rangle \\
& (+j, k | (2 \leq j \leq n) \wedge (j = k + 1) : -a[k]) \\
= & \langle (3.84)(a) \rangle \\
& (+j, k | (2 \leq k + 1 \leq n) \wedge (j = k + 1) : -a[k]) \\
= & \langle \text{Arithmetic} \rangle \\
& (+j, k | (1 \leq k \leq n - 1) \wedge (j = k + 1) : -a[k]) \\
= & \langle (+j, k | R : P) = (+k, j | R : P) \rangle \\
& (+k, j | (1 \leq k \leq n - 1) \wedge (j = k + 1) : -a[k]) \\
= & \langle (8.20), j \text{ d.n.o.f. in } 1 \leq k \leq n - 1 \rangle \\
& (+k | 1 \leq k \leq n - 1 : (+j | j = k + 1 : -a[k])) \\
= & \langle (8.14), j \text{ d.n.o.f. in } k + 1 \rangle \\
& (+k | 1 \leq k \leq n - 1 : -a[k]) \\
= & \langle (8.21), k \text{ d.n.o.f. in } 1 \leq j \leq n - 1, -a[j] \rangle \\
& (+j | 1 \leq j \leq n - 1 : -a[j])
\end{aligned}$$

- (b) Prove  $\vdash (+j | R : 0) = 0$ .  
Hint:  $0 = (+k | false : 0)$ .

**Answer:** Choose  $k$  to be fresh.

$$\begin{aligned}
& (+j \mid R : 0) \\
= & \langle (8.13) \rangle \\
& (+j \mid R : (+k \mid false : 0)) \\
= & \langle (8.20), k \text{ d.n.o.f. in } R \rangle \\
& (+j, k \mid R \wedge false : 0) \\
= & \langle (3.40) \rangle \\
& (+j, k \mid false : 0) \\
= & \langle (8.20), k \text{ d.n.o.f. in } false \rangle \\
& (+j \mid false : (+k \mid : 0)) \\
= & \langle (8.13) \rangle \\
& 0
\end{aligned}$$

(c) Rewrite using the notation of Chapter 8 of the text and prove :

$$\vdash \sum_{j=1}^n (a[j] - a[j-1]) = a[n] - a[0] .$$

You may use standard arithmetic “facts” such as  $a - b = a + (-b)$ . You may find the results in (a) and (b) useful.

**Answer:** In the notation of the text we must prove

$$\vdash (+j \mid 1 \leq j \leq n : a[j] - a[j-1]) = a[n] - a[0] .$$

$$\begin{aligned}
& (+j \mid 1 \leq j \leq n : a[j] - a[j-1]) \\
= & \langle \text{Arithmetic} \rangle \\
& (+j \mid 1 \leq j \leq n : a[j] + (-a[j-1])) \\
= & \langle (8.15) \rangle \\
& (+j \mid 1 \leq j \leq n : a[j]) + (+j \mid 1 \leq j \leq n : -a[j-1]) \\
= & \langle (8.16) \text{ twice} \rangle \\
& (+j \mid 1 \leq j \leq n-1 : a[j]) + (+j \mid j = n : a[j]) \\
& + (+j \mid j = 1 : -a[j-1]) + (+j \mid 2 \leq j \leq n : -a[j-1]) \\
= & \langle (8.14) \text{ twice, } j \text{ d.n.o.f. in } n, j \text{ d.n.o.f. in } 1 \rangle \\
& (+j \mid 1 \leq j \leq n-1 : a[j]) + a[n] + (-a[0]) + (+j \mid 2 \leq j \leq n : -a[j-1]) \\
= & \langle \text{Arithmetic, Part (a)} \rangle \\
& a[n] - a[0] + (+j \mid 1 \leq j \leq n-1 : a[j]) + (+j \mid 1 \leq j \leq n-1 : -a[j]) \\
= & \langle (8.15) \rangle \\
& a[n] - a[0] + (+j \mid 1 \leq j \leq n-1 : a[j] + (-a[j])) \\
= & \langle \text{Arithmetic} \rangle \\
& a[n] - a[0] + (+j \mid 1 \leq j \leq n-1 : 0)
\end{aligned}$$

$$\begin{aligned}
&= \langle \text{Part (b)} \rangle \\
&\quad a[n] - a[0] + 0 \\
&= \langle \text{Arithmetic} \rangle \\
&\quad a[n] - a[0]
\end{aligned}$$

5. Use the identity  $k^2 - (k-1)^2 = 2k - 1$  and a sum between 1 and  $n$  to obtain a theorem from which one can prove

$$\vdash (+k|1 \leq k \leq n : 2k - 1) = n^2$$

Hint: Use the result from Problem 4(c).

**Answer:**

$$\begin{aligned}
&\quad (+k|1 \leq k \leq n : 2k - 1) \\
&= \langle 2k - 1 = k^2 - (k-1)^2 \rangle \\
&\quad (+k|1 \leq k \leq n : k^2 - (k-1)^2) \\
&= \langle \text{Problem 4(c)} \rangle \\
&\quad n^2 - 0^2 \\
&= \langle \text{Arithmetic} \rangle \\
&\quad n^2
\end{aligned}$$

6. (D. Knuth) Find a simple formula for

$$\prod_{j=2}^n \left(1 - \frac{1}{j^2}\right)$$

and *sketch* a proof that your answer is correct.

**Answer:** We have

$$\prod_{j=2}^n \left(1 - \frac{1}{j^2}\right) = \frac{n+1}{2n}.$$

Using conventional notation,

$$\begin{aligned}
&\prod_{j=2}^n \left(1 - \frac{1}{j^2}\right) \\
&= \prod_{j=2}^n \frac{j^2 - 1}{j^2} \\
&= \prod_{j=2}^n \frac{(j-1)(j+1)}{j^2} \\
&= \prod_{j=2}^n (j-1) \prod_{j=2}^n \left(\frac{1}{j}\right) \cdot \prod_{j=2}^n \left(\frac{1}{j}\right) \prod_{j=2}^n (j+1) \\
&= \frac{1}{n} \cdot \frac{n+1}{2}
\end{aligned}$$

$$= \frac{n+1}{2n}.$$

7. (a) (Multivariate (9.2)): Prove,  $\vdash (\forall x, y | R : P) \equiv (\forall x, y | : R \Rightarrow P)$ .

**Answer:**

$$\begin{aligned} & (\forall x, y | R : P) \\ = & \langle (8.20) \rangle \\ & (\forall x | : (\forall y | R : P)) \\ = & \langle (9.2) \rangle \\ & (\forall x | : (\forall y | : R \Rightarrow P)) \\ = & \langle (8.20) \rangle \\ & (\forall x, y | : R \Rightarrow P) \end{aligned}$$

- (b) (Multivariate (8.13) where  $\star$  is  $\vee$ ): Prove,  $\vdash (\exists x, y | false : P) \equiv false$ .

**Answer:**

$$\begin{aligned} & (\exists x, y | false : P) \\ = & \langle (8.20) \rangle \\ & (\exists x | false : (\exists y | : P)) \\ = & \langle (8.13) \rangle \\ & false \end{aligned}$$

- (c) (Multivariate (9.16)): Prove, if  $\vdash (\forall x, y | : P)$ , then  $\vdash P$ .

**Answer:** By (8.20), we have  $\vdash (\forall x, y | : P) \equiv (\forall x | : (\forall y | : P))$ . Since  $\vdash (\forall x, y | : P)$ , by Equanimity,  $\vdash (\forall x | : (\forall y | : P))$ . By (9.16),  $\vdash (\forall y | : P)$ . A second application of (9.16) gives  $\vdash P$ .

- (d) (Multivariate (9.5)): Prove that provided there are no free occurrences of  $x$  or  $y$  in  $P$ ,

$$\vdash P \vee (\forall x, y | R : Q) \equiv (\forall x, y | R : P \vee Q).$$

**Answer:**

$$\begin{aligned} & P \vee (\forall x, y | R : Q) \\ = & \langle (8.20) \rangle \\ & P \vee (\forall x | : (\forall y | R : Q)) \\ = & \langle (9.5), x \text{ d.n.o.f. } P \rangle \\ & (\forall x | : P \vee (\forall y | R : Q)) \\ = & \langle (9.5), y \text{ d.n.o.f. } P \rangle \\ & (\forall x | : (\forall y | R : P \vee Q)) \\ = & \langle (8.20) \rangle \\ & (\forall x, y | R : P \vee Q) \end{aligned}$$