A fourth variation of a theme from Dow

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Jonathon Ostroff recently sent me four calculational proofs of this theorem: $S \subset T \Rightarrow T \neq \emptyset$. All four proofs used the formal definitions of $\subset$ and $\subseteq$ and therefore required the use of quantification. Below is a proof that does not require quantification. It relies solely on the theorems of chapter 3 (propositional calculus) and chapter 10 (set theory) of the Gries/Schneider text, *A Logical Approch to Discrete Math*. The theorem numbers are from that text.

$$
\begin{array}{ll}
 & S \subset T \ \Rightarrow \ T \neq \emptyset \\
= & \quad \langle \text{definition of } \subset \ (11.14) \rangle \\
 & S \subseteq T \ \wedge \ S \neq T \ \Rightarrow \ T \neq \emptyset \\
= & \quad \langle \text{Shunting (3.65); Contrapositive (3.61), Shunting (3.65)} \rangle \\
 & S \subseteq T \ \wedge \ T = \emptyset \ \Rightarrow \ S = T \\
= & \quad \langle \text{Substitution (3.84)} \rangle \\
 & S \subseteq \emptyset \ \wedge \ T = \emptyset \ \Rightarrow \ S = T \\
= & \quad \langle \text{Lemma 1 (see below)} \rangle \\
 & S = \emptyset \ \wedge \ T = \emptyset \ \Rightarrow \ S = T \ \text{ —Transitivity of } =
\end{array}
$$

Lemma 1. $S \subseteq \emptyset \ \equiv \ S = \emptyset$.

$$
\begin{array}{ll}
 & S = \emptyset \\
= & \quad \langle \text{Antisymmetry of } \subseteq \ (11.57) \rangle \\
 & S \subseteq \emptyset \ \wedge \ \emptyset \subseteq S \\
= & \quad \langle \ \emptyset \subseteq S \ \ (11.60); \text{ any theorem equivales } \mathit{true} \ \rangle \\
 & S \subseteq \emptyset \ \wedge \ \mathit{true} \\
= & \quad \langle \text{Identity of } \wedge \ (3.39) \rangle \\
 & S \subseteq \emptyset
\end{array}
$$

When it is possible, it is best to stay away quantification, since predicate calculus is more complicated than propositional calculus. Because the operations of set theory are defined in terms of quantification, some proofs must use quantification, but once a certain level reached, i.e. enough theorems have been proved, it may be possible to restrict oneself to propositional connectives.

The Gries/Schneider text does not do a proper job with the empty set. It is defined in a bulleted section on page 197 as the set $\{x \mid \mathit{false} : x\}$, which is okay. But then there is not a sufficient set of theorems concerning the empty set, such as Lemma 1 above. The next edition is expected to rectify this inadequacy.