Chapter 11 Exercise List
SOLUTIONS

1. Prove,

(a) $\vdash\ x \in \{y\} \ \equiv\ x = y\,.$
Answer:

$$
\begin{array}{ll}
 & x \in \{y\} \\
= & \langle \text{ Notation } \rangle \\
 & x \in \{x \,|\, x = y\} \\
= & \langle \text{ (11.7) } \rangle \\
 & x = y\ .
\end{array}
$$

(b) $\vdash\ \{u\} = \{x, y\} \ \Rightarrow\ x = y\,.$
Answer: Take v, w to be fresh variables.

$$
\begin{array}{ll}
 & \{u\} = \{x, y\} \\
= & \langle \text{ Notation, (11.4), } v \text{ d.n.o.f. in } \{u\}, \{x, y\} \ \rangle \\
 & (\forall v \,|\, : v \in \{u\} \ \equiv\ v \in \{w \,|\, w = x \vee w = y : w\}) \\
= & \langle \text{ 4.(c), (11.3), } w \text{ d.n.o.f. in } u \ \rangle \\
 & (\forall v \,|\, : v = u \ \equiv\ (\exists w \,|\, w = x \vee w = y : v = w)\,) \\
= & \langle \text{ (8.18), (8.14) twice } \rangle \\
 & (\forall v \,|\, : v = u \ \equiv\ (v = x \vee v = y)\,) \\
= & \langle \text{ (9.13) and (3.60) twice } \rangle \\
 & (\forall v \,|\, : v = u \ \equiv\ (v = x \vee v = y)\,) \ \wedge\ (x = u \ \equiv\ (x = x \vee x = y)\,) \\
 & \wedge\, (y = u \ \equiv\ (y = x \vee y = y)\,) \\
\Rightarrow & \langle \text{ (3.76)(b) } \rangle \\
 & (x = u \ \equiv\ (x = x \vee x = y)\,) \ \wedge\ (y = u \ \equiv\ (y = x \vee y = y)\,) \\
= & \langle \text{ (1.2) and (3.29) twice } \rangle \\
 & (x = u \ \equiv\ \mathit{true}) \ \wedge\ (y = u \ \equiv\ \mathit{true}) \\
= & \langle \text{ (3.3) twice } \rangle \\
 & x = u \ \wedge\ y = u \\
= & \langle \text{ (3.84)(a) } \rangle \\
 & x = y \wedge y = u \\
\Rightarrow & \langle \text{ (3.76)(b) } \rangle \\
 & x = y\ .
\end{array}
$$

(c) $\vdash\ \{x, y\} = \{x, z\} \ \Rightarrow\ y = z\ .$
Answer: Take w to be a fresh variable.

$$\{x, y\} = \{x, z\}$$

$$
\begin{aligned}
= & \ \langle\ (11.4), \text{Notation}\ \rangle \\
& (\forall w \,|\, : w \in \{w \,|\, w = x \ \vee\ w = y\} \ \equiv\ w \in \{w \,|\, w = x \ \vee\ w = z\}) \\
= & \ \langle\ (11.7)\ \rangle \\
& (\forall w \,|\, : w = x \ \vee\ w = y \ \equiv\ w = x \ \vee\ w = z) \\
= & \ \langle\ (9.13), (3.60)\ \rangle \\
& (\forall w \,|\, : w = x \ \vee\ w = y \ \equiv\ w = x \ \vee\ w = z) \wedge (y = x \ \vee\ y = y \ \equiv\ y = x \ \vee\ y = z) \\
& \wedge (z = x \ \vee\ z = y \ \equiv\ z = x \ \vee\ z = z) \\
\Rightarrow & \ \langle\ (3.76)(b)\ \rangle \\
& (y = x \ \vee\ y = z) \ \wedge\ (z = x \ \vee\ y = z) \\
= & \ \langle\ (3.45)\ \rangle \\
& (y = z) \ \vee\ (y = x \ \wedge\ z = x) \\
= & \ \langle\ (1.4), (3.60)\ \rangle \\
& (y = z) \ \vee\ (y = x \ \wedge\ z = x \ \wedge\ y = z) \\
= & \ \langle\ (3.43)\ \rangle \\
& y = z
\end{aligned}
$$

2. (a) Prove, provided x does not have free occurrences in S or T,

$$\vdash\ S \neq T \ \equiv\ (\exists x \,|\, x \in S : x \notin T) \ \vee\ (\exists x \,|\, x \in T : x \notin S)\ .$$

Answer:

$$
\begin{aligned}
& S \neq T \\
= & \ \langle\ \text{Notation}\ \rangle \\
& \neg(S = T) \\
= & \ \langle\ (11.4),\ x \text{ d.n.o.f. } S, T\ \rangle \\
& \neg(\forall x \,|\, : x \in S \ \equiv\ x \in T) \\
= & \ \langle\ (3.80)\ \rangle \\
& \neg(\forall x \,|\, : (x \in S \Rightarrow x \in T) \ \wedge\ (x \in T \Rightarrow x \in S)) \\
= & \ \langle\ (8.15)\ \rangle \\
& \neg((\forall x \,|\, : x \in S \Rightarrow x \in T) \ \wedge\ (\forall x \,|\, : x \in T \Rightarrow x \in S)) \\
= & \ \langle\ (9.2)\ \rangle \\
& \neg((\forall x \,|\, x \in S : x \in T) \ \wedge\ (\forall x \,|\, x \in T : x \in S)) \\
= & \ \langle\ (3.47)(a)\ \rangle \\
& \neg(\forall x \,|\, x \in S : x \in T) \ \vee\ \neg(\forall x \,|\, x \in T : x \in S) \\
= & \ \langle\ 9.18(c)\ \rangle \\
& (\exists x \,|\, x \in S : x \notin T) \ \vee\ (\exists x \,|\, x \in T : x \notin S)
\end{aligned}
$$

(b) Prove,

$$\vdash\ S \neq T \ \equiv\ S \cap T \subset S \cup T\ .$$

Answer: Take x to be a fresh variable.

$$
\begin{aligned}
& S \cap T \subset S \cup T \\
= \ & \langle \ (11.14) \ \rangle \\
& S \cap T \subseteq S \cup T \ \wedge \ S \cap T \neq S \cup T \\
= \ & \langle \ \text{Lemma: } \vdash \ S \cap T \ \subseteq S \cup T \ \rangle \\
& \mathit{true} \ \wedge \ S \cap T \neq S \cup T \\
= \ & \langle \ (3.39) \ \rangle \\
& S \cap T \neq S \cup T \\
= \ & \langle \ \text{Part (a)} \ \rangle \\
& (\exists x \,|\, x \in S \cap T : x \notin S \cup T) \ \vee \ (\exists x \,|\, x \in S \cup T : x \notin S \cap T) \\
= \ & \langle \ \text{Lemma: } \vdash \neg(\exists x \,|\, x \in S \cap T : x \notin S \cup T) \ \rangle \\
& \mathit{false} \ \vee \ (\exists x \,|\, x \in S \cup T : x \notin S \cap T) \\
= \ & \langle \ (3.30) \ \rangle \\
& (\exists x \,|\, x \in S \cup T : x \notin S \cap T) \\
= \ & \langle \ (11.20), (11.21), (3.47)(a) \ \rangle \\
& (\exists x \,|\, x \in S \ \vee x \in T : x \notin S \ \vee \ x \notin T) \\
= \ & \langle \ (9.19) \ \rangle \\
& (\exists x \,| \ : (x \in S \ \vee x \in T) \ \wedge \ (x \notin S \ \vee \ x \notin T)) \\
= \ & \langle \ (3.46) \ \rangle \\
& (\exists x \,| \ : (x \in S \wedge x \notin S) \vee (x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S) \vee (x \in T \wedge x \notin T)) \\
= \ & \langle \ (3.42), (3.30) \ \rangle \\
& (\exists x \,| \ : (x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S)) \\
= \ & \langle \ (8.15) \ \rangle \\
& (\exists x \,| \ : x \in S \wedge x \notin T) \ \vee (\exists x \,| \ : x \in T \wedge x \notin S) \\
= \ & \langle \ (9.19) \ \rangle \\
& (\exists x \,|\, x \in S : x \notin T) \ \vee (\exists x \,|\, x \in T : x \notin S) \\
= \ & \langle \ \text{Part (a)} \ \rangle \\
& S \neq T
\end{aligned}
$$

3. (a) Prove, $\vdash \ S = T \ \equiv \ S \subseteq T \ \wedge \ T \subseteq S$.
Answer: Take z to be a fresh variable.

$$
\begin{aligned}
& S = T \\
= \ & \langle \ (11.4), \ z \text{ d.n.o.f. } S, T \ \rangle \\
& (\forall z \,| \ : z \in S \ \equiv \ z \in T) \\
= \ & \langle \ (3.80) \ \rangle \\
& (\forall z \,| \ : (z \in S \ \Rightarrow \ z \in T) \ \wedge (z \in T \ \Rightarrow \ z \in S)) \\
= \ & \langle \ (8.15) \ \rangle \\
& (\forall z \,| \ : z \in S \ \Rightarrow \ z \in T) \ \wedge (\forall z \,| \ : z \in T \ \Rightarrow \ z \in S)
\end{aligned}
$$

$$
\begin{aligned}
&= \langle\ (9.2)\ \rangle \\
&\quad (\forall z \mid z \in S : z \in T)\ \wedge (\forall z \mid z \in T : z \in S) \\
&= \langle\ (11.13),\ z \text{ d.n.o.f. } S, T\ \rangle \\
&\quad S \subseteq T\ \wedge\ T \subseteq S
\end{aligned}
$$

(b) Prove, $\vdash\ S \subset T \ \Rightarrow\ T \not\subseteq S$.
Hint: $\vdash\ P \Rightarrow Q\ \equiv\ P \wedge \neg Q\ \equiv\ false$.
Answer: By the hint it suffices to prove

$$\vdash S \subset T\ \wedge\ \neg(T \not\subseteq S)\ \ \equiv\ \ false\ .$$

$$
\begin{aligned}
&\quad S \subset T\ \wedge\ \neg(T \not\subseteq S) \\
&= \langle\ (3.12)\ \rangle \\
&\quad S \subset T\ \wedge\ T \subseteq S \\
&= \langle\ (11.14)\ \rangle \\
&\quad S \subseteq T\ \wedge\ S \neq T\ \wedge\ T \subseteq S \\
&= \langle\ \text{Part (a)}\ \rangle \\
&\quad S = T\ \wedge\ S \neq T \\
&= \langle\ (3.42)\ \rangle \\
&\quad false
\end{aligned}
$$

4. (a) Prove, $\vdash\ \{y\} \subseteq \{x \mid R\}\ \ \equiv\ \ R[x := y]$.
Answer: Choose z to be a fresh variable.

$$
\begin{aligned}
&\quad \{y\} \subseteq \{x \mid R\} \\
&= \langle\ (11.13),\ z \text{ d.n.o.f. } \{y\}, \{x \mid R\}\ \rangle \\
&\quad (\forall z \mid z \in \{y\} : z \in \{x \mid R\}) \\
&= \langle\ \text{Notation, Dummy renaming for set comprehensions, } z \text{ d.n.o.f. } x,\ R\ \rangle \\
&\quad (\forall z \mid z \in \{z | z = y\} : z \in \{z \mid R[x := z]) \\
&= \langle\ (11.7)\ \rangle \\
&\quad (\forall z \mid z = y : R[x := z]) \\
&= \langle\ (8.14)\ \rangle \\
&\quad R[x := z][z := y] \\
&= \langle\ \text{Recursive definition of contextual substitution, (3.5)}\ \rangle \\
&\quad R[x := y]
\end{aligned}
$$

(b) Prove, $\vdash\ S \subset \{x \mid R\}\ \ \Rightarrow\ \ (\exists x \mid : R)$.
Answer: By Contrapositive it suffices to prove $\vdash\ (\forall x \mid : \neg R)\ \Rightarrow\ S \not\subset \{x \mid R\}$. Assume

$(\forall x \mid : \neg R)$. Note that the variable x does not occur free in the assumption. By (9.13) and Strong Modus Ponens, $\neg R$ is a temporary theorem.

$$
\begin{array}{ll}
& S \not\subset \{x|R\} \\
= & \langle\ \neg R,\ x \text{ d.n.o.f. Assumption }\rangle \\
& S \not\subset \{x|false\} \\
= & \langle\ \text{Definition of } \emptyset\ \rangle \\
& S \not\subset \emptyset \\
= & \langle\ (11.14)\ \rangle \\
& \neg(S \subseteq \emptyset \wedge S \neq \emptyset) \\
= & \langle\ \text{Lemma: } \vdash\ S \subseteq \emptyset\ \equiv\ S = \emptyset\ \rangle \\
& \neg(S = \emptyset \wedge S \neq \emptyset) \\
= & \langle\ (3.42)\ \rangle \\
& \neg false \\
= & \langle\ (3.13)\ \rangle \\
& true
\end{array}
$$

5. (a) Prove, $\vdash\ \{x \mid false : E) \ = \ \{x \mid false : F)$.
   Answer: Take z to be a fresh variable.

$$
\begin{array}{ll}
& \{x \mid false : E\} \ = \ \{x \mid false : F\} \\
= & \langle\ (11.4),\ z \text{ d.n.o.f. } \{x \mid false : E\}, \{x \mid false : F\}\ \rangle \\
& (\forall z \mid\ : z \in \{x \mid false : E\} \ \equiv \ z \in \{x \mid false : F\}) \\
= & \langle\ (11.3),\ x \text{ d.n.o.f. } z\ \rangle \\
& (\forall z \mid\ : (\exists x \mid false : z = E) \ \equiv \ (\exists x \mid false : x = F)) \\
= & \langle\ (8.13)\ \rangle \\
& (\forall z \mid\ : false \equiv false) \\
= & \langle\ (3.3)\ \rangle \\
& (\forall z \mid\ : true) \\
= & \langle\ (9.8)\ \rangle \\
& true
\end{array}
$$

   (b) Prove, $\vdash\ z \in \{x \mid false : E\} \ \equiv \ false$.
   Answer: Take y to be a fresh variable.

$$
\begin{array}{ll}
& z \in \{x \mid false : E\} \\
= & \langle\ \text{Dummy renaming for set comprehensions}\ \rangle \\
& z \in \{y \mid false : E[x := y]\} \\
= & \langle\ (11.3),\ y \text{ d.n.o.f. in } z\ \rangle \\
& (\exists y \mid false : z = E[x := y])
\end{array}
$$

$$
\begin{aligned}
&= \langle\ (8.13)\ \rangle \\
&\quad \mathit{false}\ .
\end{aligned}
$$

(c) Prove that provided z d.n.o.f. in S or E,

$$\vdash\ S = \{x \mid \mathit{false} : E) \ \equiv\ (\forall z \mid \ : z \notin S)\ .$$

Answer:

$$
\begin{aligned}
&\quad S = \{x \mid \mathit{false} : E\} \\
&= \langle\ (11.4),\ z \text{ d.n.o.f. } S, \{x \mid \mathit{false} : E\}\ \rangle \\
&\quad (\forall z \mid \ : z \in S \ \equiv\ z \in \{x \mid \mathit{false} : E)) \\
&= \langle\ (11.3),\ x \text{ d.n.o.f. } z\ \rangle \\
&\quad (\forall z \mid \ : z \in S \ \equiv\ (\exists x \mid \mathit{false} : z = E)) \\
&= \langle\ (8.13)\ \rangle \\
&\quad (\forall z \mid \ : z \in S \ \equiv\ \mathit{false}) \\
&= \langle\ (3.15)\ \rangle \\
&\quad (\forall z \mid \ : z \notin S)
\end{aligned}
$$

(d) Prove that $\quad \vdash\ \ S \subset \{x\}\ \ \Rightarrow\ \ S = \emptyset\ .$

Answer: Take z to be a fresh variable. We have

$$
\begin{aligned}
&\quad S \subset \{x\} \ \Rightarrow\ S = \emptyset \\
&= \langle\ \text{Lemma: } \vdash S = \emptyset \ \equiv\ (\forall z \mid \ : z \notin S)\ \rangle \\
&\quad S \subset \{x\} \ \Rightarrow\ (\forall z \mid \ : z \notin S) \\
&= \langle\ (3.61),\ (9.17)\ \rangle \\
&\quad (\exists z \mid \ : z \in S) \ \Rightarrow\ s \not\subset \{x\}
\end{aligned}
$$

Take w to be a fresh variable. By (9.30), it suffices to prove that $\vdash\ w \in S \ \Rightarrow\ \neg(S \subset \{x\}$.

$$
\begin{aligned}
&\quad w \in S \ \Rightarrow\ \neg(S \subset \{x\}) \\
&= \langle\ (3.59)\ \rangle \\
&\quad \neg(w \in S) \ \vee\ \neg(S \subset \{x\}) \\
&= \langle\ (3.47)(a)\ \rangle \\
&\quad \neg(w \in S \ \wedge\ S \subset \{x\}) \\
&= \langle\ (11.14)\ \rangle \\
&\quad \neg(w \in S \ \wedge\ S \subseteq \{x\} \ \wedge\ S \neq \{x\}) \\
&= \langle\ \text{Lemma: } \vdash w \in S \wedge S \subseteq \{x\} \ \Rightarrow\ w \in \{x\},\ (3.60)\ \rangle \\
&\quad \neg(w \in S \ \wedge\ S \subseteq \{x\} \ \wedge\ w \in \{x\} \ \wedge\ S \neq \{x\}) \\
&= \langle\ \text{Problem 1(a)}\ \rangle
\end{aligned}
$$

$$
\begin{aligned}
& \neg(w \in S \ \wedge\ S \subseteq \{x\} \ \wedge\ w = x \ \wedge\ S \neq \{x\}) \\
= \ & \langle\ (3.84)(a)\ \rangle \\
& \neg(w \in S \ \wedge\ S \subseteq \{w\} \ \wedge\ w = x \ \wedge\ S \neq \{w\}) \\
= \ & \langle\ \text{Lemma: } \vdash\ w \in S \ \Rightarrow\ \{w\} \subseteq S,\ (3.60)\ \rangle \\
& \neg(w \in S \ \wedge\ \{w\} \subseteq S \ \wedge\ S \subseteq \{w\} \ \wedge\ w = x \ \wedge\ S \neq \{w\}) \\
= \ & \langle\ (11.57)\ \rangle \\
& \neg(w \in S \ \wedge\ S = \{w\} \ \wedge\ w = x \ \wedge\ S \neq \{w\}) \\
= \ & \langle\ (3.42)\ \rangle \\
& \neg(w \in S \ \wedge\ w = x \ \wedge\ false) \\
= \ & \langle\ (3.40)\ \rangle \\
& \neg false \\
= \ & \langle\ (3.13)\ \rangle \\
& true
\end{aligned}
$$

6. (a) Prove that $\vdash\ (\exists y| : (\exists x| : S = \{x, y\}\ )\ )\ \Rightarrow\ \#S \leq 1 + 1$.

(If you wish, in this problem you may use the following :

1. For any ("finite") boolean expression R , $\vdash\ (\sum z|R : 1) \geq 0$.
2. $\vdash\ B \geq 0 \wedge B + C = D\ \Rightarrow\ C \leq D$.)

Answer: Take z and w to be fresh variables. By (9.30) for multiple quantifications, it suffices to prove $\vdash\ S = \{z, w\}\ \Rightarrow\ \#S \leq 1 + 1$. Take u to be a fresh variable. Assume $S = \{z, w\}$. We are using an extended proof form which exploits transitivity of $=$ and $\leq$ for type $\mathbb{N}$.

$$
\begin{aligned}
& \#S \\
= \ & \langle\ (11.12),\ u \text{ d.n.o.f. } S\ \rangle \\
& (+u \,|\, u \in S : 1) \\
= \ & \langle\ \text{Assumption, } u \text{ d.n.o.f. Assumption}\ \rangle \\
& (+u \,|\, u \in \{z, w\} : 1) \\
= \ & \langle\ \text{Lemma: } \vdash\ u \in \{z, w\}\ \equiv\ u = z \vee u = w\ \rangle \\
& (+u \,|\, u = z \ \vee\ u = w : 1) \\
= \ & \langle\ (8.17)\ \rangle \\
& (+u \,|\, u = z : 1) + (+u \,|\, u = w : 1) - (+u \,|\, u = v \ \wedge\ u = w : 1) \\
\leq \ & \langle\ \text{Arithmetic, Facts above}\ \rangle \\
& (+u \,|\, u = z : 1) + (+u \,|\, u = w : 1) \\
= \ & \langle\ (8.14)\ \rangle \\
& 1 + 1
\end{aligned}
$$

(b) Is $(\exists y| : (\exists x| : S = \{x, y\}))\ \Rightarrow\ \#S = 1 + 1$ a theorem?
If so, prove it.

If not, explain in detail why it is not.
Answer: It is not a theorem. Consider $S = \{z\} = \{z, z\}$. By (9.28) for multiple quantifications, $\vdash S = \{z, z\} \;\Rightarrow\; (\exists y| :\; (\exists x| :\; S = \{x, y\}))$ but $\#S = 1$.

7. Interpret the symbols 1 , 3 , and $+$ in the usual way. Is there an interpretation such that

$$(+x|\; x \in \{x, y\} :\; 1) \;=\; 3$$

is satisfied (i.e., interprets as True)? If so, give a universe of discourse etc. for such an interpretation. If not, explain in detail why not.
Answer: Yes. Take as universe of discourse $\{0, 1, 2\}$ or any other set with 3 elements. As $x \in \{x, y\}$ is True, the expression $(+x|\; x \in \{x, y\} :\; 1)$ has value the number of elements in the universe of discourse.

8. (a) Prove, $\vdash S \subseteq T \;\equiv\; \mathcal{P}S \subseteq \mathcal{P}T$.
Answer: Take y to be a fresh variable.

$$
\begin{aligned}
& \mathcal{P}S \subseteq \mathcal{P}T \\
= \;& \langle\; (11.13) \;\rangle \\
& (\forall y \,|\, y \in \mathcal{P}S : y \in \mathcal{P}T) \\
= \;& \langle\; (11.23) \;\rangle \\
& (\forall y \,|\, y \subseteq S : y \subseteq T) \\
= \;& \langle\; (9.2) \;\rangle \\
& (\forall y \,|\; : y \subseteq S \Rightarrow y \subseteq T) \\
= \;& \langle\; (9.13), (3.60) \;\rangle \\
& (\forall y \,|\; : y \subseteq S \Rightarrow y \subseteq T) \;\wedge\; (S \subseteq S \Rightarrow S \subseteq T) \\
= \;& \langle\; (11.58), (3.73) \;\rangle \\
& (\forall y \,|\; : y \subseteq S \Rightarrow y \subseteq T) \;\wedge\; (S \subseteq T) \\
= \;& \langle\; (9.7), \text{Lemma: } \vdash \neg(\forall x \,|\; : \neg true), \text{Modus Ponens} \;\rangle \\
& (\forall y \,|\; : S \subseteq T \;\wedge\; (y \subseteq S \Rightarrow y \subseteq T)) \\
= \;& \langle\; \text{Lemma: } \vdash S \subseteq T \;\Rightarrow (y \subseteq S \;\Rightarrow\; y \subseteq T), (3.60) \;\rangle \\
& (\forall y \,|\; : S \subseteq T) \\
= \;& \langle\; (3.39), (9.7), \text{Lemma: } \vdash \neg(\forall x \,|\; : \neg true), \text{Modus Ponens} \;\rangle \\
& (\forall y \,|\; : true) \;\wedge\; S \subseteq T \\
= \;& \langle\; (9.8), (3.39) \;\rangle \\
& S \subseteq T
\end{aligned}
$$

(b) Prove, $\vdash S \neq \emptyset \;\;\Rightarrow\;\; \mathcal{P}S \neq \{\emptyset\}$.
Answer: Take x to be a fresh variable.

$$
\begin{aligned}
& S \neq \emptyset \;\Rightarrow\; \mathcal{P}S \neq \{\emptyset\} \\
= \;& \langle\; \text{Definition of } \emptyset \;\rangle \\
& (\exists x \,|\; : x \in S) \;\Rightarrow\; \mathcal{P}S \neq \{\emptyset\}
\end{aligned}
$$

Take w and y to be fresh variables. By (9.30) it suffices to prove $\vdash\ w \in S\ \Rightarrow\ \mathcal{P}S \neq \{\emptyset\}$.

$$
\begin{array}{ll}
 & w \in S \\
\Rightarrow & \langle\ \text{Lemma: } \vdash w \in S\ \Rightarrow \{w\} \subseteq S\ \rangle \\
 & \{w\} \subseteq S \\
= & \langle\ (11.23)\ \rangle \\
 & \{w\} \in \mathcal{P}S \\
= & \langle\ \text{Lemma: } \{s\} \neq \emptyset\ \rangle \\
 & \{w\} \in \mathcal{P}S\ \wedge \{w\} \neq \emptyset \\
\Rightarrow & \langle\ (9.28)\ \rangle \\
 & (\exists y \mid\ : y \in \mathcal{P}S \wedge y \neq \emptyset) \\
= & \langle\ (9.19)\ \rangle \\
 & (\exists y \mid y \in \mathcal{P}S : y \neq \emptyset) \\
\Rightarrow & \langle\ \text{Lemma: } \vdash\ T = \{\emptyset\}\ \Rightarrow\ (\forall y \mid y \in T : y = \emptyset),\ (3.61)\ \rangle \\
 & \mathcal{P}S \neq \{\emptyset\}
\end{array}
$$

9. (a) Prove, $\vdash\ (\forall z \mid\ : \neg R)\ \Rightarrow\ (+z \mid R : 1) = 0$.
Answer: Assume $(\forall z \mid\ : \neg R)$, from which it follows that $R \equiv \mathit{false}$.

$$
\begin{array}{ll}
 & (+z \mid R : 1)\ =\ 0 \\
= & \langle\ \text{Strong Leibniz, } z \text{ d.n.o.f. in } (\forall z \mid\ : \neg R)\ \rangle \\
 & (+z \mid \mathit{false} : 1)\ =\ 0
\end{array}
$$

(b) Prove, $\vdash\ \#\{x,y\} = \#\{y\}\ \Rightarrow\ x = y$.
Answer: Choose z to be a fresh variable.

$$
\begin{array}{ll}
 & \#\{x,y\} = \#\{y\} \\
= & \langle\ (11.12),\ z \text{ d.n.o.f. } \{x,y\},\ \{y\}\ \rangle \\
 & (+z \mid z \in \{x,y\} : 1) = (+z \mid z \in \{y\} : 1) \\
= & \langle\ \text{Notation}\ \rangle \\
 & (+z \mid z \in \{z \mid z = x\ \vee\ z = y\} : 1) = (+z \mid z \in \{z \mid z = y\} : 1) \\
= & \langle\ (11.7)\ \rangle \\
 & (+z \mid z = x\ \vee\ z = y : 1) = (+z \mid z = y : 1) \\
= & \langle\ (8.17),\ \text{Algebra}\ \rangle \\
 & (+z \mid z = x : 1) + (+z \mid z = y : 1) - (+z \mid z = x\ \wedge\ z = y : 1) = (+z \mid z = y : 1) \\
= & \langle\ \text{Algebra, } (8.14)\ \rangle \\
 & (+z \mid z = x\ \wedge\ z = y : 1) = 1 \\
\Rightarrow & \langle\ \vdash 1 \neq 0,\ \text{Part (a), } (3.61)\ \rangle \\
 & (\exists z \mid\ : z = x\ \wedge\ z = y) \\
= & \langle\ (3.84)(\text{a})\ \rangle
\end{array}
$$

$$
\begin{array}{ll}
 & (\exists z \mid \, : z = x \ \wedge \ x = y) \\
= & \langle \ (9.21) \ \rangle \\
 & (\exists z \mid \, : z = x) \ \wedge \ x = y \\
\Rightarrow & \langle \ (3.76)(\text{b}) \ \rangle \\
 & x = y
\end{array}
$$

10. (a) Prove, $\vdash \ w \in S \ \ \Rightarrow \ \ S = (S - \{w\}) \cup \{w\}$.

Answer: Assume $w \in S$.

Let x be a fresh variable.

We will prove $x \in (S - \{w\}) \cup \{w\} \equiv x \in S$.

$$
\begin{array}{ll}
 & x \in (S - \{w\}) \cup \{w\} \ \ \equiv \ \ x \in S \\
= & \quad \langle \ (11.20) \ \rangle \\
 & x \in (S - \{w\}) \ \vee \ x \in \{w\} \ \ \equiv \ \ x \in S \\
= & \quad \langle \ (11.22) \ \rangle \\
 & (x \in S \ \wedge \ x \notin \{w\}) \ \vee \ x \in \{w\} \ \ \equiv \ \ x \in S \\
= & \quad \langle \ (3.45) \ \rangle \\
 & (x \in S \ \vee \ x \in \{w\}) \wedge (x \notin \{w\} \ \vee x \in \{w\}) \ \ \equiv \ \ x \in S \\
= & \quad \langle \ (3.28) \text{ Metatheorem: If } \vdash P \text{ then } \vdash P \equiv \mathit{true} \, . \ \rangle \\
 & (x \in S \ \vee \ x \in \{w\}) \wedge \ \mathit{true} \ \ \equiv \ \ x \in S \\
= & \quad \langle \ (3.39) \ \rangle \\
 & (x \in S \ \vee \ x \in \{w\}) \ \ \equiv \ \ x \in S \\
= & \quad \langle \ (3.57) \ \rangle \\
 & x \in \{w\} \Rightarrow x \in S \\
= & \quad \langle \ \text{Part (a)} \ \rangle \\
 & x = w \ \Rightarrow \ x \in S \\
= & \quad \langle \ (3.84)(\text{b}) \ \rangle \\
 & x = w \ \Rightarrow \ w \in S \\
= & \quad \langle \ \text{Assumption} \ \rangle \\
 & x = w \ \Rightarrow \ \mathit{true} \\
= & \quad \langle \ (3.72) \ \rangle \\
 & \mathit{true}
\end{array}
$$

(b) i. Prove, $\vdash \ \#S = 0 \ \ \equiv \ \ S = \emptyset$.

Answer: Proceed by Mutual Implication. First prove

$$S = \emptyset \ \Rightarrow \ \#S = 0$$

using the Deduction Theorem.

Assume $S = \emptyset$. Take x to be a fresh variable. Then

$$
\begin{array}{ll}
 & \#S = 0 \\
= & \quad \langle \, (11.12) \, \rangle \\
 & (+x | \, x \in S : 1) = 0 \\
= & \quad \langle \, \text{Assumption} \, \rangle \\
 & (+x | \, \mathit{false} : 1) = 0 \\
= & \quad \langle \, (8.13) \, \rangle \\
 & 0 = 0 \ .
\end{array}
$$

We prove

$$\#S = 0 \Rightarrow S = \emptyset$$

by contrapositive, i.e., by proving

$$S \neq \emptyset \;\Rightarrow\; \#S \neq 0.$$

For x a fresh variable

$$S \neq \emptyset \;\equiv\; (\exists x \,|\, : x \in S)$$

so that we can prove

$$(\exists x \,|\, : x \in S) \;\Rightarrow\; \#S \neq 0$$

or by Metatheorem Witness, where w is taken to be a fresh variable,

$$w \in S \;\Rightarrow\; \#S \neq 0\ .$$

Assume $w \in S$.

$\quad \#S \neq 0$
$= \quad \langle\ (11.12)\ \rangle$
$\quad (+x \,|\, x \in S : 1) \neq 0$
$= \quad \langle$ Assumption, Question 3(a), Strong Modus Ponens $\rangle$
$\quad (+x \,|\, x \in (S - \{w\}) \cup \{w\} : 1) \neq 0$
$= \quad \langle\ (11.20)\ \rangle$
$\quad (+x \,|\, x \in (S - \{w\}) \vee x \in \{w\} : 1) \neq 0$
$= \quad \langle\ (8.16)\ \rangle$
$\quad (+x \,|\, x \in S - \{w\} : 1) + (+x \,|\, x \in \{w\} : 1) \neq 0$
$= \quad \langle$ Question 1(a) $\rangle$
$\quad (+x \,|\, x \in S - \{w\} : 1) + (+x \,|\, x = w : 1) \neq 0$
$= \quad \langle\ (8.14)\ \rangle$
$\quad (+x \,|\, x \in S - \{w\} : 1) + 1 \neq 0$

Domain reasoning gives that the last line is a theorem. Cardinality is a non-negative integer.

ii. Prove, $\vdash\ \#S = 1 \;\equiv\; (\exists x \,|\, : S = \{x\})$.

Answer: Proceed by Mutual Implication. To prove

$$(\exists x \,|\, : S = \{x\}) \Rightarrow \#S = 1$$

it suffices by Metatheorem Witness to prove

$$S = \{w\} \Rightarrow \#S = 1\ .$$

Assume $S = \{w\}$. Take z to be a fresh variable.

$\quad \#S = 1$
$= \quad \langle\ (11.12)\ \rangle$
$\quad (+z| \, z \in S : 1) = 1$
$= \quad \langle$ Assumption $\rangle$
$\quad (+z| \, z \in \{w\} : 1) = 1$
$= \quad \langle$ Question 1(a) $\rangle$
$\quad (+z| \, z = w : 1) = 1$
$= \quad \langle\ (8.14)\ \rangle$
$\quad 1 = 1$

To prove

$$\#S = 1 \Rightarrow (\exists x \mid : S = \{x\})$$

observe that by (i), $\#S = 1 \;\Rightarrow\; S \neq \emptyset$, so that $\#S = 1 \;\Rightarrow\; (\exists z \mid : z \in S)$.
We will prove

$$\#S = 1 \wedge (\exists z \mid : z \in S) \;\Rightarrow\; (\exists x \mid : S = \{x\}))\ .$$

This follows if we prove

$$(\exists z \mid : z \in S) \;\Rightarrow\; (\#S = 1 \;\Rightarrow\; (\exists x \mid : S = \{x\})$$

or by Metatheorem Witness with w taken to be a fresh variable,

$$w \in S \;\Rightarrow\; (\#S = 1 \;\Rightarrow\; (\exists x \mid : S = \{x\}))\ .$$

Assume $w \in S$. By Strong Modus Ponens $S = (S - \{s\}) \cup \{s\}$, from which by (8.16) $\#S = \#(S - \{s\}) + \#\{s\}$.
Assume $\#S = 1$. Then

$$
\begin{array}{ll}
 & \#S = \#(S - \{w\}) + \#\{w\} \\
= & \quad \langle \text{ Previous part of proof } \rangle \\
 & 1 = \#(S - \{w\}) + 1 \\
= & \quad \langle \text{ Arithmetic } \rangle \\
 & \#(S - \{w\}) = 0 \\
= & \quad \langle \text{ Previous part } \rangle \\
 & S - \{w\} = \emptyset
\end{array}
$$

Finally,

$$
\begin{array}{ll}
 & S \\
= & \quad \langle \text{ Question 3(a) } \rangle \\
 & (S - \{w\}) \cup \{w\} \\
= & \quad \langle \text{ Previous part of proof } \rangle \\
 & \emptyset \cup \{w\} \\
= & \quad \langle \text{ (11.30) } \rangle \\
 & \{w\}\ .
\end{array}
$$

By (9.28) we have

$$\vdash\ S = \{w\} \quad \Rightarrow \quad (\exists x \mid : S = \{x\})$$

which by Strong Modus Ponens we gives $(\exists x \mid\ : S = \{x\})$ as required.

11. The following series of exercises identifies $\mathcal{P}\{E, F\}$ as $\{\emptyset, \{E\}, \{F\}.\{E, F\}\}$.

    (a) Prove

$$\vdash\ E \in S\ \wedge\ F \in S \quad \equiv \quad \{E, F\} \subseteq S$$

Answer: Assume $E \in S$. Assume $F \in S$. Let z be a fresh variable.

$$
\begin{array}{ll}
 & \{E, F\} \subseteq S \\
= & \Big\langle \text{ (11.13) } \Big\rangle \\
 & (\forall z | z \in \{E, F\} : z \in S) \\
= & \Big\langle \text{ Notation, (9.2) } \Big\rangle \\
 & (\forall z | : z \in \{z \mid z = E\ \vee\ z = F\} \;\Rightarrow\; z \in S)
\end{array}
$$

$$
\begin{array}{cl}
= & \langle\ (11.7)\ \rangle \\
& (\forall z| : z = E \ \vee\ z = F \ \Rightarrow\ z \in S) \\
= & \langle\ \text{Lemma} \vdash\ (P \vee Q \Rightarrow R)\ \equiv\ (P \Rightarrow R) \wedge (Q \Rightarrow R)\ \rangle \\
& (\forall z| : (z = E \ \Rightarrow\ z \in S)\ \wedge\ (z = F \ \Rightarrow\ z \in S)\,) \\
= & \langle\ (3.84)(\text{b})\ \rangle \\
& (\forall z| : (z = E \ \Rightarrow\ E \in S)\ \wedge\ (z = F \ \Rightarrow\ F \in S)\,) \\
= & \langle\ \text{Assumption}\ \rangle \\
& (\forall z| : (z = E \ \Rightarrow\ \mathit{true})\ \wedge\ (z = F \ \Rightarrow\ \mathit{true})\,) \\
= & \langle\ (3.72)\ \rangle \\
& (\forall z| : \mathit{true}\ \wedge\ \mathit{true}) \\
= & \langle\ (3.38)\ \rangle \\
& (\forall z| : \mathit{true}) \\
= & \langle\ (9.8)\ \rangle \\
& \mathit{true}\ .
\end{array}
$$

(b) Prove

$$\vdash\ S \subseteq \{E, F\}\ \wedge\ F \notin S \ \ \Rightarrow\ \ S \subseteq \{E\}\ .$$

Answer: Let z be a fresh variable.

$$
\begin{array}{cl}
& S \subseteq \{E, F\}\ \wedge\ F \notin S \\
= & \langle\ (11.13)\ \rangle \\
& (\forall z\,|\,z \in S : z \in \{E, F\})\ \wedge\ F \notin S \\
= & \langle\ (11.3)\ \rangle \\
& (\forall z\,|\,z \in S : z \in \{E, F\})\ \wedge\ \neg(\exists z\,|\,z \in S : F = z) \\
= & \langle\ \text{Notation, (1.2), (9.18)(c)}\ \rangle \\
& (\forall z\,|\,z \in S : z \in \{z\,|\,z = E\ \vee\ z = F\})\ \wedge\ (\forall z\,|\,z \in S : z \neq F) \\
= & \langle\ \text{(11.7), (9.2) twice}\ \rangle \\
& (\forall z|\ : z \in S \ \Rightarrow\ z = E\ \vee\ z = F)\ \wedge\ (\forall z|\ : z \in S \ \Rightarrow\ z \neq F) \\
= & \langle\ (8.15)\ \rangle \\
& (\forall z|\ : (z \in S \ \Rightarrow\ z = E\ \vee\ z = F)\ \wedge\ (z \in S \ \Rightarrow\ z \neq F)\,) \\
= & \langle\ \text{Lemma} \vdash\ (P \Rightarrow R) \wedge (P \Rightarrow Q)\ \equiv\ (P \Rightarrow Q \wedge R)\ \rangle \\
& (\forall z|\ : z \in S \ \Rightarrow\ (z = E\ \vee\ z = F)\ \wedge\ (z \neq F)\,) \\
= & \langle\ \text{(3.46), (3.42), (3.40)}\ \rangle \\
& (\forall z|\ : z \in S \ \Rightarrow\ z = E\ \wedge\ z \neq F) \\
= & \langle\ \ \text{Lemma} \vdash\ (P \Rightarrow Q \wedge R)\ \Rightarrow\ (P \Rightarrow Q)\text{, (9.12), Modus Ponens}\ \rangle \\
& (\forall z|\ : z \in S \ \Rightarrow\ z = E) \\
= & \langle\ (9.2)\ \rangle \\
& (\forall z\,|\,z \in S : z = E)
\end{array}
$$

$$
\begin{aligned}
= \quad & \langle \text{ Lemma} \vdash z = E \; \equiv \; z \in \{E\} \; \rangle \\
& (\forall z \,|\, z \in S : z \in \{E\}) \\
= \quad & \langle \; (11.13) \; \rangle \\
& S \subseteq \{E\}
\end{aligned}
$$

(c) Prove

$$\vdash w \in S \; \wedge \; S \subseteq T \quad \Rightarrow \quad w \in T \; .$$

Answer:
Choose x to be a fresh variable.

$$
\begin{aligned}
& w \in S \; \wedge \; S \subseteq T \\
= \quad & \langle \; (11.13) \; \rangle \\
& w \in S \; \wedge \; (\forall x \,|\, x \in S : x \in T) \\
= \quad & \langle \; (9.2) \; \rangle \\
& w \in S \; \wedge \; (\forall x \,|\; : x \in S \Rightarrow x \in T) \\
= \quad & \langle \; (9.13), \; (3.60) \; \rangle \\
& w \in S \; \wedge \; (w \in S \Rightarrow w \in T) \; \wedge \; (\forall x \,|\; : x \in S \Rightarrow x \in T) \\
= \quad & \langle \; (3.66) \; \rangle \\
& w \in S \; \wedge \; w \in T \; \wedge \; (\forall x \,|\; : x \in S \Rightarrow x \in T) \\
\Rightarrow \quad & \langle \; (3.76)(\text{b}) \; \rangle \\
& w \in T \; .
\end{aligned}
$$

(d) Prove

$$\vdash S \subseteq \{E, F\} \; \wedge \; w \in S \quad \Rightarrow \quad w = E \; \vee \; w = F \; .$$

Answer:

$$
\begin{aligned}
& S \subseteq \{E, F\} \; \wedge \; w \in S \\
\Rightarrow \quad & \langle \text{ Part (c) } \rangle \\
& w \in \{E, F\} \\
= \quad & \langle \text{ Notation } \rangle \\
& w \in \{w \,|\, w = E \vee w = F\} \\
= \quad & \langle \; (11.7) \; \rangle \\
& w = E \; \vee \; w = F \; .
\end{aligned}
$$

(e) Prove

$$\vdash S \subseteq \{E, F\} \; \wedge \; (\exists x \,|\; : x \in S) \quad \Rightarrow \quad E \in S \; \vee \; F \in S \; .$$

Hint: Start by using Metatheorem Witness.

Answer: Take w to be a fresh variable. By (3.65) and (9.30) it suffices to prove

$$\vdash S \subseteq \{E, F\} \wedge w \in S \Rightarrow E \in S \vee F \in S .$$

$$\begin{array}{ll}
 & S \subseteq \{E, F\} \wedge w \in S \\
= & \langle \text{ Part (d), (3.60) } \rangle \\
 & (S \subseteq \{E, F\} \wedge w \in S) \wedge (w = E \vee w = F) \\
= & \langle \text{ (3.46) } \rangle \\
 & (S \subseteq \{E, F\} \wedge w \in S \wedge w = E) \vee (S \subseteq \{E, F\} \wedge w \in S \wedge w = F) \\
= & \langle \text{ (3.84)(a) twice } \rangle \\
 & (S \subseteq \{E, F\} \wedge E \in S \wedge w = E) \vee (S \subseteq \{E, F\} \wedge F \in S \wedge w = F) \\
\Rightarrow & \langle \text{ (3.76)(d) twice } \rangle \\
 & E \in S \vee F \in S .
\end{array}$$

(f) Prove $\mathcal{P}\{E, F\} = \{\emptyset, \{E\}, \{F\}, \{E, F\}\}$.
Answer:
Hint: Prove

$$\vdash S \subseteq \{E, F\} \wedge (\exists x \mid : x \in s) \Rightarrow S = \{E\} \vee S = \{F\} \vee S = \{E, F\}$$

using Case Analysis. The condition $E \in S \vee F \in S$ from Part (e) gives three cases.
Answer: As

$$\vdash S \subseteq \{E, F\} \wedge (\exists x \mid x \in s) \Rightarrow E \in S \vee F \in S ,$$

it follows that

$$\vdash S \subseteq \{E, F\} \wedge (\exists x \mid x \in s) \equiv S \subseteq \{E, F\} \wedge (\exists x \mid x \in s) \wedge (E \in S \vee F \in S) .$$

We consider three cases.

$$S \subseteq \{E, F\} \wedge (\exists x \mid : x \in S) \wedge (E \in S \wedge F \in S) \Rightarrow S = \{E\} \vee S = \{F\} \vee S = \{E, F\} .$$
$$S \subseteq \{E, F\} \wedge (\exists x \mid : x \in S) \wedge (E \in S \wedge F \notin S) \Rightarrow S = \{E\} \vee S = \{F\} \vee S = \{E, F\} .$$
$$S \subseteq \{E, F\} \wedge (\exists x \mid : x \in S) \wedge (E \notin S \wedge F \in S) \Rightarrow S = \{E\} \vee S = \{F\} \vee S = \{E, F\} .$$

$$\begin{array}{ll}
 & S \subseteq \{E, F\} \wedge (\exists x \mid : x \in S) \wedge (E \in S \wedge F \in S) \\
\Rightarrow & \langle \text{ (3.76)(b) } \rangle \\
 & S \subseteq \{E, F\} \wedge E \in S \wedge F \in S \\
= & \langle \text{ Part (a) } \rangle \\
 & S \subseteq \{E, F\} \wedge \{E, F\} \subseteq S \\
= & \langle \text{ (11.57) } \rangle \\
 & S = \{E, F\} \\
\Rightarrow & \langle \text{ (3.76)(a) } \rangle \\
 & S = \{E\} \vee S = \{F\} \vee S = \{E, F\} .
\end{array}$$

$$
\begin{aligned}
& S \subseteq \{E, F\} \ \wedge\ (\exists x \mid\ : x \in S) \ \wedge\ (E \in S \ \wedge\ F \notin S) \\
\Rightarrow\ & \langle\ (3.76)(b)\ \rangle \\
& S \subseteq \{E, F\} \ \wedge\ E \in S \ \wedge\ F \notin S \\
=\ & \langle\ (3.36)\ \rangle \\
& S \subseteq \{E, F\} \ \wedge\ F \notin S \ \wedge\ E \in S \\
\Rightarrow\ & \langle\ \text{Part (b), (4.3), Modus Ponens}\ \rangle \\
& S \subseteq \{E\} \ \wedge\ E \in S \\
=\ & \langle\ \text{Lemma} \vdash E \in S \equiv \{E\} \subseteq S\ \rangle \\
& S \subseteq \{E\} \ \wedge\ \{E\} \subseteq S \\
=\ & \langle\ (11.57)\ \rangle \\
& S = \{E\} \\
\Rightarrow\ & \langle\ (3.76)(a)\ \rangle \\
& S = \{E\} \ \vee\ S = \{F\} \ \vee\ S = \{E, F\}\ .
\end{aligned}
$$

The proof for Case 3 is identical.