# Chapter 11 Exercise List

1. Prove,

   (a) $\vdash\ x \in \{y\} \quad \equiv \quad x = y\,.$

   (b) $\vdash\ \{u\} = \{x, y\} \quad \Rightarrow \quad x = y\,.$

   (c) $\vdash\ \{x, y\} = \{x, z\} \quad \Rightarrow \quad y = z\ .$

2. (a) Prove, provided $x$ does not have free occurrences in $S$ or $T$,

$$\vdash\ S \neq T \quad \equiv \quad (\exists x \,|\, x \in S : x \notin T) \ \vee\ (\exists x \mid x \in T : x \notin S)\ .$$

   (b) Prove,

$$\vdash\ S \neq T \quad \equiv \quad S \cap T \subset S \cup T\ .$$

3. (a) Prove, $\vdash\ S = T \quad \equiv \quad S \subseteq T \ \wedge\ T \subseteq S.$

   (b) Prove, $\vdash\ S \subset T \quad \Rightarrow \quad T \not\subseteq S.$
   Hint: $\vdash\ P \Rightarrow Q \ \equiv\ P \wedge \neg Q \ \equiv\ false.$

4. (a) Prove, $\vdash\ \{y\} \subseteq \{x \,|\, R\} \quad \equiv \quad R[x := y]\ .$

   (b) Prove, $\vdash\ S \subset \{x \,|\, R\} \quad \Rightarrow \quad (\exists x \,|\, : R)\ .$

5. (a) Prove, $\vdash\ \{x \,|\, false : E) \quad = \quad \{x \,|\, false : F)\ .$

   (b) Prove, $\vdash\ z \in \{x \,|\, false : E\} \quad \equiv \quad false\ .$

   (c) Prove that provided $z$ d.n.o.f. in $S$ or $E$,

$$\vdash\ S = \{x \,|\, false : E) \quad \equiv \quad (\forall z \,|\ : z \notin S)\ .$$

   (d) Prove that $\quad \vdash \quad S \subset \{x\} \quad \Rightarrow \quad S = \emptyset\ .$

6. (a) Prove that $\quad \vdash \quad (\exists y |\, :\ (\exists x |\, :\ S = \{x, y\}\ )\ ) \quad \Rightarrow \quad \#S \leq 1 + 1\ .$
   (If you wish, in this problem you may use the following :
   1. For any ("finite") boolean expression $R$ , $\vdash$ $(\sum z | R : 1) \geq 0$ .
   2. $\vdash \quad B \geq 0 \ \wedge\ B + C = D \quad \Rightarrow \quad C \leq D$ .)

   (b) Is $\ (\exists y |\, :\ (\exists x |\, :\ S = \{x, y\})) \quad \Rightarrow \quad \#S = 1 + 1\ $ a theorem?
   If so, prove it.
   If not, explain in detail why it is not.

7. Interpret the symbols 1 , 3 , and + in the usual way. Is there an interpretation such that

$$(+x |\ x \in \{x, y\} :\ 1) \ = \ 3$$

   is satisfied (i.e., interprets as True)? If so, give a universe of discourse etc. for such an interpretation. If not, explain in detail why not.

8. (a) Prove, $\vdash\ S \subseteq T \quad \equiv \quad \mathcal{P}S \subseteq \mathcal{P}T\ .$

   (b) Prove, $\vdash\ S \neq \emptyset \quad \Rightarrow \quad \mathcal{P}S \neq \{\emptyset\}\ .$

9. (a) Prove, $\vdash\ (\forall z \,|\, : \neg R) \quad \Rightarrow \quad (+z \,|\, R : 1) = 0\ .$

(b) Prove, $\vdash \ \#\{x,y\} = \#\{y\} \quad \Rightarrow \quad x = y$ .

10. (a) Prove, $\vdash \ w \in S \quad \Rightarrow \quad S = (S - \{w\}) \cup \{w\}$ .
    (b) i. Prove, $\vdash \ \#S = 0 \quad \equiv \quad S = \emptyset$ .
        ii. Prove, $\vdash \ \#S = 1 \quad \equiv \quad (\exists x \mid \, : S = \{x\})$ .

11. The following series of exercises identifies $\mathcal{P}\{E,F\}$ as $\{\emptyset, \{E\}, \{F\}.\{E,F\}\}$.

    (a) Prove,
$$\vdash \ E \in y \ \wedge \ F \in y \quad \Rightarrow \quad \{E,F\} \subseteq y \ .$$
    (b) Prove,
$$\vdash \ y \subseteq \{E,F\} \ \wedge \ F \notin y \quad \Rightarrow \quad y \subseteq \{E\} \ .$$
    (c) Prove,
$$\vdash \ w \in y \ \wedge y \subseteq z \quad \Rightarrow \quad w \in z \ .$$
    (d) Prove,
$$\vdash \ y \subseteq \{E,F\} \ \wedge \ w \in y \quad \Rightarrow \quad w = E \ \vee \ w = F \ .$$
    (e) Prove,
$$\vdash \ y \subseteq \{E,F\} \ \wedge \ (\exists x \mid \, : x \in y) \quad \Rightarrow \quad E \in y \ \vee \ F \in y \ .$$
    (f) Prove,
$$\vdash \ y \subseteq \{E,F\} \ \wedge \ (\exists x \mid \, : x \in y) \quad \Rightarrow \quad y = \{E\} \ \vee \ y = \{F\} \ \vee \ y = \{E,F\} \ .$$

    **Hint:** Use **Case Analysis**. The condition, $E \in y \ \vee \ F \in y$, in (e) gives three cases.