Chapter 12 Exercise List
SOLUTIONS

1. Let P be an expression of type Boolean. Suppose n dnof in P . Let

$$x_1,\ x_2,\ x_3,\ \ldots,\ x_n,\ x_{n+1},\ \ldots$$

be an infinite sequence of variables that do not occur freely in P .
Define boolean expressions $E.j$ inductively (recursively) by defining

$E.0$ to be P , and
$E(n+1)$ to be $(\forall x_{n+1} | :\ E.n)$.

(Examples: $E.2$ could be $(\forall x | :\ (\forall y | :\ P))$;
$E.3$ could be $(\forall z | :\ (\forall x | :\ (\forall y | :\ P)))$, etc.)

(a) Prove by Mathematical Induction that $\vdash\ (\forall n : \mathbb{N} | :\ P \Rightarrow E.n)$.
Answer: Note that $P.n$ is $P \Rightarrow E.n$ and that in particular, $P.0$ is $P \Rightarrow E.0$.
For the base case, we have

$$\begin{array}{cl} & P \Rightarrow E.0 \\ = & \langle \text{ Recursive Definition } \rangle \\ & P \Rightarrow P \\ = & \langle \text{ (3.71) } \rangle \\ & \mathit{true} \end{array}$$

For the Induction Step, assume $P.n$, i.e., $P \Rightarrow E.n$.

$$\begin{array}{cl} & P \Rightarrow E.(n+1) \\ = & \langle \text{ Recursive Definition } \rangle \\ & P \Rightarrow (\forall x_{n+1} \mid\ : E.n) \\ = & \langle \text{ (3.59) } \rangle \\ & \neg P \vee (\forall x_{n+1} \mid\ : E.n) \\ = & \langle \text{ (9.5) } \rangle \\ & (\forall x_{n+1} \mid\ : \neg P \vee E.n) \\ = & \langle \text{ (3.59) } \rangle \\ & (\forall x_{n+1} \mid\ : P \Rightarrow E.n) \end{array}$$

which is a theorem by Strong (9.16).

(b) Give a complete, careful, syntactic proof that $\vdash\ P \Rightarrow E.143$.
(You may assume that 143 is a constant of type $\mathbb{N}$.)
Answer:
By (9.13), we have $\vdash\ (\forall n \mid P.n)\ \Rightarrow\ P.143$.
By Part (a), $\vdash\ (\forall n \mid\ : P.n)$.
By Modus Ponens, we have $\vdash$, $P.143$, i.e., $\vdash\ P \Rightarrow E.143$.

2. For sets S_i, $i \in \mathbb{N}$, define $\bigcap\limits_{i=0}^{n} S_i$ recursively by

$$\bigcap_{i=0}^{0} S_i = S_0 \text{ and } \bigcap_{i=0}^{n+1} S_i = (\bigcap_{i=0}^{n} S_i) \cap S_{n+1} \ .$$

Prove by Mathematical Induction that

$$\left(\forall n \,|\, : \bigcap_{i=0}^{n} (T_i \cap V) = (\bigcap_{i=0}^{n} T_i) \cap V \right) \ .$$

Answer: Base Case:
$\vdash \left(\forall n \,|\, : \bigcap\limits_{i=0}^{0} (T_i \cap V) = (\bigcap\limits_{i=0}^{0} T_i) \cap V \right)$.

$$\begin{aligned} & \bigcap_{i=0}^{0} (T_i \cap V) \ = \ (\bigcap_{i=0}^{0} T_i) \cap V \\ = \ & \langle \text{ Definition } \rangle \\ & T_0 \cap V \ = \ T_0 \cap V \ . \end{aligned}$$

Induction Step:
$\vdash \left(\forall j \,|\, 1 \le j \le n : \bigcap\limits_{i=0}^{j} (T_i \cap V) = (\bigcap\limits_{i=0}^{j} T_i) \cap V \right) \ \Rightarrow \ \bigcap\limits_{i=0}^{n} (T_i \cap V) = (\bigcap\limits_{i=0}^{n} T_i) \cap V$.
Assume $\left(\forall j \,|\, 1 \le j \le n : \bigcap\limits_{i=0}^{j} (T_i \cap V) = (\bigcap\limits_{i=0}^{j} T_i) \cap V \right)$ from which $\bigcap\limits_{i=0}^{n} (T_i \cap V) = (\bigcap\limits_{i=0}^{n} T_i) \cap V$.
We are using = for **type** set in the left margin.

$$\begin{aligned} & \bigcap_{i=0}^{n+1} (T_i \cap V) \\ = \ & \langle \text{ Recursive definition } \rangle \\ & \bigcap_{i=0}^{n} (T_i \cap V) \ \cap \ (T_{n+1} \cap V) \\ = \ & \langle \text{ Assumption } \rangle \\ & (\bigcap_{i=0}^{n} T_i) \cap V \ \cap \ T_{n+1} \cap V \\ = \ & \langle \text{ (11.33), (11.34), (11.35) } \rangle \\ & (\bigcap_{i=0}^{n} T_i) \ \cap \ T_{n+1} \cap V \\ = \ & \langle \text{ Recursive definition } \rangle \\ & (\bigcap_{i=0}^{n+1} T_i) \cap V \end{aligned}$$

3. This exercise is from Rosen, **Discrete Mathematics and Its Applications**. Its solution is nontrivial.

$$\text{Prove,} \quad \sum_{\emptyset \subset \{a_1,\ldots,a_k\} \subseteq \{1,2,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} \;=\; n.$$

Take x, y, z to be fresh variables. For $y \subseteq \{1, \ldots, n\}$ and $y \neq \emptyset$, define

$$\text{frac}.y = \frac{1}{(\prod x \mid x \in y : x)} \ .$$

Use Mathematical Induction to prove

$$\vdash\ (\forall n \mid n \geq 1 : (+y \mid y \subseteq \{1, \ldots, n\}\ \wedge\ y \neq \emptyset : \text{frac}.y) = n)\ .$$

Hint: Look at how one proceeds from Case $n = 2$ to Case $n = 3$. The nonempty subsets of $\{1, 2, 3\}$ are $\{1\}, \{2\}, \{1, 2\}$ which are the nonempty subsets of $\{1, 2\}$, as well as $\{3\}$, and $\{1, 3\}, \{2, 3\}, \{1, 2, 3\}$ which are those subsets of $\{1, 2, 3\}$ which are the union of nonempty subsets of $\{1, 2\}$ with $\{3\}$. If we know $\frac{1}{1} + \frac{1}{2} + \frac{1}{1\cdot 2} = 2$, then

$$\begin{aligned}
& \frac{1}{1} + \frac{1}{2} + \frac{1}{1\cdot 2} + \frac{1}{3} + \frac{1}{1\cdot 3} + \frac{1}{2\cdot 3} + \frac{1}{1\cdot 2\cdot 3} \\
= \ & \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{1\cdot 2}\right) + \left(\frac{1}{3}\right) + \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{1\cdot 2}\right) \cdot \frac{1}{3} \\
= \ & 2 + \frac{1}{3} + 2 \cdot \frac{1}{3} = 2 + 1 = 3\ .
\end{aligned}$$

The central step in the general case uses this change of dummy Lemma,

$$\begin{aligned}
\vdash\ & (+y \mid y \subseteq \{1, \ldots, n+1\}\ \wedge\ n+1 \in y\ \wedge\ y \neq \{n+1\} : \text{frac}.y) \\
& = (+z \mid z \subseteq \{1, \ldots, n\} \wedge\ z \neq \emptyset : \frac{1}{n+1}\,\text{frac}.z)\ .
\end{aligned}$$

Answer: Take $x,\ y,\ z$ to be fresh variables.
The proof is by Mathematical Induction. The base case is $n = 1$. Recall from discussion in class of $\mathcal{P}\{E\}$, that

$$\vdash\ y \subseteq \{1\}\ \wedge\ y \neq \emptyset\ \equiv\ y = \{1\}\ .$$

Now

$$\begin{aligned}
& (+y \mid y \subseteq \{1\}\ \wedge\ y \neq \emptyset : \text{frac}.y)\ =\ 1 \\
= \ & \langle\ \vdash\ y \subseteq \{1\} \wedge y \neq \emptyset\ \ \equiv\ \ y = \{1\}\ \rangle \\
& (+y \mid y = \{1\} : \text{frac}.y)\ =\ 1 \\
= \ & \langle\ (8.14)\ \rangle \\
& \text{frac}.\{1\}\ =\ 1 \\
= \ & \langle\ \text{Definition of frac}.y\ \rangle \\
& \frac{1}{(\prod x \mid x \in \{1\} : x)}\ =\ 1 \\
= \ & \langle\ \text{Lemma: } \vdash\ x \in \{1\}\ \equiv\ x = 1\ \rangle
\end{aligned}$$

$$
\begin{array}{cl}
 & \dfrac{1}{(\prod x \,|\, x = 1 : x)} \;=\; 1 \\
= & \langle\ (8.14)\ \rangle \\
 & \dfrac{1}{1} \;=\; 1\ .
\end{array}
$$

For the induction step we prove

$$\vdash\ (+y \,|\, y \subseteq \{1, \ldots, n\} \wedge y \neq \emptyset : \text{frac}.y) \;=\; n \;\;\Rightarrow\;\; (+y \,|\, y \subseteq \{1, \ldots, n{+}1\} \wedge y \neq \emptyset : \text{frac}.y) \;=\; n{+}1\ .$$

Assume $(+y \,|\, y \subseteq \{1, \ldots, n\} \;\wedge\; y \neq \emptyset : \text{frac}.y) \;=\; n$. Leibniz (8.12) may be used for quantifications over y and z as neither of these variables appear free in the assumption. Observe that under this assumption

$$
\begin{array}{cl}
 & (+y \,|\, y \subseteq \{1, \ldots, n+1\} \;\wedge\; y \neq \emptyset : \text{frac}.y) \;=\; n+1 \\
= & \langle\ \vdash P \;\equiv\; (P \wedge \neg Q) \vee (P \wedge Q)\ \rangle \\
 & (+y \,|\, (y \subseteq \{1, \ldots, n+1\} \;\wedge\; n+1 \notin y \;\wedge\; y \neq \emptyset) \\
 & \quad \vee\, (y \subseteq \{1, \ldots, n+1\} \;\wedge\; n+1 \in y \;\wedge\; y \neq \emptyset) : \text{frac}.y) \quad\;=\; n+1 \\
= & \langle\ (8.16)\ \rangle \\
 & (+y \,|\, y \subseteq \{1, \ldots, n+1\} \;\wedge\; n+1 \notin y \;\wedge\; y \neq \emptyset : \text{frac}.y) \\
 & \quad +\, (+y \,|\, y \subseteq \{1, \ldots, n+1\} \;\wedge\; n+1 \in y \;\wedge\; y \neq \emptyset : \text{frac}.y) \;\;=\;\; n+1 \\
= & \langle\ \text{Lemma: } \vdash y \subseteq \{1, \ldots, n+1\} \;\wedge\; n+1 \notin y \;\wedge\; y \neq \emptyset \;\;\equiv\;\; y \subseteq \{1, \ldots, n\} \;\wedge\; y \neq \emptyset\ \rangle \\
 & (+y \,|\, y \subseteq \{1, \ldots, n\} \;\wedge\; y \neq \emptyset : \text{frac}.y) \\
 & \quad +\, (+y \,|\, y \subseteq \{1, \ldots, n+1\} \;\wedge\; n+1 \in y \;\wedge\; y \neq \emptyset : \text{frac}.y) \;\;=\;\; n+1 \\
= & \langle\ \text{Assumption}\ \rangle \\
 & n \;+\; (+y \,|\, y \subseteq \{1, \ldots, n+1\} \;\wedge\; n+1 \in y \;\wedge\; y \neq \emptyset : \text{frac}.y) \;\;=\;\; n+1 \\
= & \langle\ \text{Algebra}\ \rangle \\
 & (+y \,|\, y \subseteq \{1, \ldots, n+1\} \;\wedge\; n+1 \in y \;\wedge\; y \neq \emptyset : \text{frac}.y) \;\;=\;\; 1\ .
\end{array}
$$

We'll be done once we prove

$$\vdash\ (+y \,|\, y \subseteq \{1, \ldots, n+1\} \;\wedge\; n+1 \in y \;\wedge\; y \neq \emptyset : \text{frac}.y) \;\;=\;\; 1\ .$$

A routine proof gives

$$\vdash\ y \subseteq \{1, \ldots, n{+}1\} \wedge n{+}1 \in y \wedge y \neq \emptyset \;\;\equiv\;\; y = \{n{+}1\} \vee (y \subseteq \{1, \ldots, n{+}1\} \wedge n{+}1 \in y \wedge y \neq \{n{+}1\})\ .$$

Applying (8.16),

$$
\begin{array}{cl}
\vdash & (+y \,|\, y \subseteq \{1, \ldots, n+1\} \;\wedge\; n+1 \in y \;\wedge\; y \neq \emptyset : \text{frac}.y) \\
= & (+y \,|\, y = \{n+1\} : \text{frac}.y) \;+\; (+y \,|\, y \subseteq \{1, \ldots, n+1\} \;\wedge\; n+1 \in y \;\wedge\; y \neq \{n+1\} : \text{frac}.y)\ .
\end{array}
$$

Since by (8.14), $\vdash\ (+y \,|\, y = \{n+1\} : \text{frac}.y) = \dfrac{1}{n+1}$, we need only prove

$$
\begin{array}{cl}
\vdash & (+y \,|\, y \subseteq \{1, \ldots, n\} \;\wedge\; y \neq \emptyset : \text{frac}.y) = n \\
\Rightarrow & (+y \,|\, y \subseteq \{1, \ldots, n+1\} \;\wedge\; n+1 \in y \;\wedge\; y \neq \{n+1\} : \text{frac}.y) = \dfrac{n}{n+1}\ .
\end{array}
$$

The result follows easily from

$$\begin{aligned}&\vdash\ (+y\,|\,y\subseteq\{1,\ldots,n+1\}\ \wedge\ n+1\in y\ \wedge\ y\neq\{n+1\}:\mathrm{frac}.y)\\ &\qquad=\ (+z\,|\,z\subseteq\{1,\ldots,n\}\wedge\ z\neq\emptyset:\frac{1}{n+1}\,\mathrm{frac}.z)\ .\end{aligned}$$

We have

$$\begin{array}{ll}
 & (+z\,|\,z\subseteq\{1,\ldots,n\}\wedge\ z\neq\emptyset:\dfrac{1}{n+1}\,\mathrm{frac}.z)\\
= & \langle\ (8.14)\ \rangle\\
 & (+z\,|\,z\subseteq\{1,\ldots,n\}\wedge\ z\neq\emptyset:(+y\,|\,y=z\cup\{n+1\}:\dfrac{1}{n+1}\,\mathrm{frac}.z))\\
= & \langle\ (8.20)\ \rangle\\
 & (+z,y\,|\,z\subseteq\{1,\ldots,n\}\ \wedge\ z\neq\emptyset\ \wedge\ y=z\cup\{n+1\}:\dfrac{1}{n+1}\,\mathrm{frac}.z)\\
= & \left\langle\begin{array}{l}\text{Lemma: }\vdash\ z\subseteq\{1,\ldots,n\}\ \wedge\ z\neq\emptyset\ \wedge\ y=z\cup\{n+1\}\\ \qquad\qquad\equiv\ y\subseteq\{1,\ldots,n+1\}\ \wedge\ n+1\in y\ \wedge\ y\neq\{n+1\}\ \wedge\ z=y-\{n+1\}\end{array}\right\rangle\\
 & (+z,y\,|\,y\subseteq\{1,\ldots,n+1\}\ \wedge\ n+1\in y\ \wedge\ y\neq\{n+1\}\ \wedge\ z=y-\{n+1\}:\dfrac{1}{n+1}\,\mathrm{frac}.z)\\
= & \langle\ \vdash\ (\star x,y\,|\,R:P)\ =\ (\star y,x\,|\,R:P)\ \rangle\\
 & (+y,z\,|\,y\subseteq\{1,\ldots,n+1\}\ \wedge\ n+1\in y\ \wedge\ y\neq\{n+1\}\ \wedge\ z=y-\{n+1\}:\dfrac{1}{n+1}\,\mathrm{frac}.z)\\
= & \langle\ (8.20)\ \rangle\\
 & (+y\,|\,y\subseteq\{1,\ldots,n+1\}\ \wedge\ n+1\in y\ \wedge\ y\neq\{n+1\}:(+z\,|\,z=y-\{n+1\}\ :\dfrac{1}{n+1}\,\mathrm{frac}.z))\\
= & \langle\ (8.14)\ \rangle\\
 & (+y\,|\,y\subseteq\{1,\ldots,n+1\}\ \wedge\ n+1\in y\ \wedge\ y\neq\{n+1\}:\dfrac{1}{n+1}\,\mathrm{frac}.(y-\{n+1\}))\\
= & \langle\ \text{Lemma: }\vdash\ n+1\in y\ \wedge\ y\neq\{n+1\}\ \Rightarrow\ \tfrac{1}{n+1}\mathrm{frac}.(y-\{n+1\})\ =\ \mathrm{frac}.y\ \rangle\\
 & (+y\,|\,y\subseteq\{1,\ldots,n+1\}\ \wedge\ n+1\in y\ \wedge\ y\neq\{n+1\}:\mathrm{frac}.y)
\end{array}$$

Exercise 8, Section 1.2.1 of Donald Knuth's *The Art of Programming, Volume 1* reads
(a) Prove the following theorem of Nicomachus (C.E. c.100) by induction:

$$1^3=1,\ 2^3=3+5,\ 3^3=7+9+11,\ 4^3=13+15+17+19,\ \text{etc.}$$

(b) Use this result to prove the remarkable formula

$$1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2\ .$$

Recall from the 1090 Review Homework,

$$\vdash\ (+k\,|\,1\leq k\leq n:2k-1)=n^2\ .$$

The following exercises provide a proof of (b) based on Knuth's approach. As in the previous problem, a change of dummy result lies at the core of the proof part (c).

4\. (a) Use Mathematical Induction to prove that provided k d.n.o.f. in P,

$$\vdash\ (+k\,|\,1 \le k \le n : P) \quad = \quad n \cdot P\ .$$

Answer:
Case $n = 0$:

$$\begin{array}{cl} & (+k\,|\,1 \le k \le 0 : P) = 0 \cdot P \\ = & \langle\ \text{Arithmetic}, \vdash\ 1 \le k \le 0\ \equiv\ \mathit{false}\ \rangle \\ & (+k\,|\,\mathit{false} : P) = 0 \\ = & \langle\ (8.13)\ \rangle \\ & 0 = 0\ . \end{array}$$

Assume $(\forall i\,|\,0 \le i \le n : (+k\,|\,1 \le k \le i : P) = i \cdot P)$ from which it follows that $(+k\,|\,1 \le k \le n : P) \quad = \quad n \cdot P$.

$$\begin{array}{cl} & (+k\,|\,1 \le k \le n+1 : P) \\ = & \langle\ (8.23)\ \rangle \\ & (+k\,|\,1 \le k \le n : P) + P[k := n+1] \\ = & \langle\ \text{Assumption, } k \text{ d.n.o.f. in } P\ \rangle \\ & n \cdot P + P \\ = & \langle\ \text{Algebra}\ \rangle \\ & (n+1) \cdot P \end{array}$$

(b) Prove

$$\vdash\ (+j\,|\,1 \le j \le n : j^3) \quad = \quad (+j\,|\,1 \le j \le n : (+k\,|\,1 \le k \le j : j^2 - j + (2k-1))\,)\ .$$

Answer:

$$\begin{array}{cl} & (+k\,|\,1 \le k \le j : j^2 - j + (2k-1)) \\ = & \langle\ (8.15)\ \rangle \\ & (+k\,|\,1 \le k \le j : j^2 - j) + (+k\,|\,1 \le k \le j : 2k-1) \\ = & \langle\ \text{Problem 5, 1090 Review}\ \rangle \\ & j \cdot (j^2 - j) + j^2 \\ = & \langle\ \text{Algebra}\ \rangle \\ & j^3\ . \end{array}$$

The result follows using Leibniz.

(c) Prove

$$\vdash\ (n+1)^2-(n+1)=2\cdot(+k\,|\,1\le k\le n:k)$$

and use Mathematical Induction to prove

$$\vdash\ (+j\,|\,1\le j\le n:(+k\,|\,1\le k\le j:j^2{-}j{+}(2k{-}1))\,)\quad=\quad(+k\,|\,1\le k\le(+j\,|\,1\le j\le n:j):2k{-}1)\ .$$

Answer:

$$\begin{array}{cl}
 & (n+1)^2-(n+1)\\
= & \langle\ \text{Quoted result from 1090 Review}\ \rangle\\
 & (+k\,|\,1\le k\le n+1:2k-1)-(n+1)\\
= & \langle\ (8.23)\ \rangle\\
 & (+k\,|\,1\le k\le n:2k-1)+(2(n+1)-1)-(n+1)\\
= & \langle\ (8.15),\ \text{Algebra}\ \rangle\\
 & (+k\,|\,1\le k\le n:2k)+(+k\,|\,1\le k\le n:-1)+n\\
= & \langle\ \text{Problem 5}\ \rangle\\
 & (+k\,|\,1\le k\le n:2k)-n+n\\
= & \langle\ \text{Algebra}\ \rangle\\
 & (+k\,|\,1\le k\le n:k+k)\\
= & \langle\ (8.15),\ \text{Algebra}\ \rangle\\
 & 2\cdot(+k\,|\,1\le k\le n:k)\ .
\end{array}$$

To prove

$$\vdash\ (+j\,|\,1\le j\le n:(+k\,|\,1\le k\le j:j^2{-}j{+}(2k{-}1))\,)\quad=\quad(+k\,|\,1\le k\le(+j\,|\,1\le j\le n:j):2k{-}1),$$

consider the Base Case, $n=0$.

$$\begin{array}{cl}
 & (+j\,|\,1\le j\le 0:(+k\,|\,1\le k\le j:j^2-j+(2k-1))\,)\\
 & \quad=\quad(+k\,|\,1\le k\le(+j\,|\,1\le j\le 0:j):2k-1)\\
= & \langle\ \text{Arithmetic},\vdash\ 1\le j\le 0\ \equiv\ \mathit{false}\ \ \rangle\\
 & (+j\,|\,\mathit{false}:(+k\,|\,1\le k\le j:j^2-j+(2k-1))\,)\\
 & \quad=\quad(+k\,|\,1\le k\le(+j\,|\,\mathit{false}:j):2k-1)\\
= & \langle\ (8.13)\ \rangle\\
 & 0\quad=\quad(+k\,|\,1\le k\le 0:2k-1)\\
= & \langle\ \ \text{Arithmetic},\vdash\ 1\le k\le 0\ \equiv\ \mathit{false}\ \rangle\\
 & 0\quad=\quad(+k\,|\,\mathit{false}:2k-1)\ .
\end{array}$$

Now assume

$$\begin{array}{l}
(\forall i\,|\,0\le i\le n:(+j\,|\,1\le j\le i:(+k\,|\,1\le k\le j:j^2-j+(2k-1))\,)\\
\qquad\qquad\qquad=\quad(+k\,|\,1\le k\le(+j\,|\,1\le j\le i:j):2k-1))\ ,
\end{array}$$

from which it follow that

$$(+j\,|\,1 \le j \le n : (+k\,|\,1 \le k \le j : j^2{-}j{+}(2k{-}1))\,) \quad = \quad (+k\,|\,1 \le k \le (+j\,|\,1 \le j \le n : j) : 2k{-}1)\ .$$

$$\begin{array}{cl} & (+j\,|\,1 \le j \le n+1 : (+k\,|\,1 \le k \le j : j^2 - j + (2k-1))\,) \\ = & \langle\ (8.23)\ \rangle \\ & (+j\,|\,1 \le j \le n : (+k\,|\,1 \le k \le j : j^2 - j + (2k-1))\,) \\ & \ \ +\ \ (+k\,|\,1 \le k \le n+1 : (n+1)^2 - (n+1) + (2k-1)\,) \\ = & \langle\ \text{previous result}\ \rangle \\ & (+k\,|\,1 \le k \le (+j\,|\,1 \le j \le n : j) : 2k-1) \\ & \ \ +\ \ (+k\,|\,1 \le k \le n+1 : 2 \cdot (+j\,|\,1 \le j \le n : j) + (2k-1)\,) \\ = & \langle\ \text{Algebra}\ \rangle \\ & (+k\,|\,1 \le k \le (+j\,|\,1 \le j \le n : j) : 2k-1) \\ & \ \ +\ \ (+k\,|\,1 \le k \le n+1 : 2 \cdot ((+j\,|\,1 \le j \le n : j) + k) - 1\,) \\ = & \langle\ \text{Change of Dummy, proof similar to Problem 4(a), 1090 Review}\ \rangle \\ & (+k\,|\,1 \le k \le (+j\,|\,1 \le j \le n : j) : 2k-1) \\ & \ \ +\ \ (+k\,|\,(+j\,|\,1 \le j \le n : j) + 1 \le k \le (+j\,|\,1 \le j \le n : j) + (n+1) : 2k-1\,) \\ = & \langle\ (8.23)\ \rangle \\ & (+k\,|\,1 \le k \le (+j\,|\,1 \le j \le n : j) : 2k-1) \\ & \ \ +\ \ (+k\,|\,(+j\,|\,1 \le j \le n : j) + 1 \le k \le (+j\,|\,1 \le j \le n+1 : j) : 2k-1\,) \\ = & \langle\ (8.16)\ \rangle \\ & (+k\,|\,1 \le k \le (+j\,|\,1 \le j \le n+1 : j) : 2k-1\,) \end{array}$$

(d) Prove

$$\vdash (+j\,|\,1 \le j \le n : j^3) \quad = \quad (+j\,|\,1 \le j \le n : j)^2\ .$$

Answer: Problems 2. and 3. give

$$\vdash\ (+j\,|\,1 \le j \le n : j^3) \quad = \quad (+k\,|\,1 \le k \le (+j\,|\,1 \le j \le n+1 : j) : 2k-1)\ .$$

But by the result from Homework 2,

$$\vdash\ (+k\,|\,1 \le k \le (+j\,|\,1 \le j \le n+1 : j) : 2k-1) = (+j\,|\,1 \le j \le n+1 : j)^2\ .$$

By (1.4),

$$\vdash (+j\,|\,1 \le j \le n : j^3) \quad = \quad (+j\,|\,1 \le j \le n : j)^2\ .$$