Chapter 12 Exercise List

1. Let P be an expression of type Boolean. Suppose n dnof in P . Let

   $$x_1,\ x_2,\ x_3,\ \ldots,\ x_n,\ x_{n+1},\ \ldots$$

   be an infinite sequence of variables that do not occur freely in P .
   Define boolean expressions $E.j$ inductively (recursively) by defining

   $E.0$ to be P , and
   $E(n+1)$ to be $(\forall x_{n+1}| :\ E.n)$.

   (Examples: $E.2$ could be $(\forall x| :\ (\forall y| :\ P))$;
   $E.3$ could be $(\forall z| :\ (\forall x| :\ (\forall y| :\ P)))$, etc.)

   (a) Prove by Mathematical Induction that $\vdash\ (\forall n : \mathbb{N}\,| :\ P \Rightarrow E.n)$.
   (b) Give a complete, careful, syntactic proof that $\vdash\ P \Rightarrow E.143$.
   (You may assume that 143 is a constant of type $\mathbb{N}$.)

2. For sets S_i, $i \in \mathbb{N}$, define $\bigcap_{i=0}^{n} S_i$ recursively by

   $$\bigcap_{i=0}^{0} S_i = S_0 \ \text{ and } \ \bigcap_{i=0}^{n+1} S_i = (\bigcap_{i=0}^{n} S_i\,) \cap S_{n+1}\ .$$

   Prove by Mathematical Induction that

   $$\left(\forall n\,|\ : \bigcap_{i=0}^{n} (T_i \cap V) = (\bigcap_{i=0}^{n} T_i\,) \cap V\right)\ .$$

3. This exercise is from Rosen, **Discrete Mathematics and Its Applications**. Its solution is nontrivial.

   $$\text{Prove,} \quad \sum_{\emptyset \subset \{a_1,\ldots,a_k\} \subseteq \{1,2,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} \ = \ n.$$

   Take x, y, z to be fresh variables. For $y \subseteq \{1,\ldots,n\}$ and $y \neq \emptyset$, define

   $$\text{frac}.y = \frac{1}{(\prod x\,|\,x \in y : x)}\ .$$

   Use Mathematical Induction to prove

   $$\vdash\ (\forall n\,|\,n \geq 1 : (+y\,|\,y \subseteq \{1,\ldots,n\}\ \wedge\ y \neq \emptyset : \text{frac}.y) = n)\ .$$

   Hint: Look at how one proceeds from Case $n = 2$ to Case $n = 3$. The nonempty subsets of $\{1,2,3\}$ are $\{1\},\{2\},\{1,2\}$ which are the nonempty subsets of $\{1,2\}$, as well as $\{3\}$, and $\{1,3\},\{2,3\},\{1,2,3\}$ which are those subsets of $\{1,2,3\}$ which are the union of nonempty subsets of $\{1,2\}$ with $\{3\}$. If we know $\frac{1}{1}+\frac{1}{2}+\frac{1}{1\cdot 2} = 2$, then

   $$\begin{aligned} & \frac{1}{1}+\frac{1}{2}+\frac{1}{1\cdot 2}+\frac{1}{3}+\frac{1}{1\cdot 3}+\frac{1}{2\cdot 3}+\frac{1}{1\cdot 2\cdot 3} \\ = & \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{1\cdot 2}\right)+\left(\frac{1}{3}\right)+\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{1\cdot 2}\right)\cdot\frac{1}{3} \\ = & \ 2+\frac{1}{3}+2\cdot\frac{1}{3} = 2+1 = 3\ . \end{aligned}$$

The central step in the general case uses this change of dummy Lemma,

$$\begin{aligned}\vdash\ & (+y \mid y \subseteq \{1,\ldots,n+1\}\ \wedge\ n+1 \in y\ \wedge\ y \neq \{n+1\} : \text{frac}.y) \\ &= (+z \mid z \subseteq \{1,\ldots,n\} \wedge\ z \neq \emptyset : \frac{1}{n+1}\,\text{frac}.z)\ .\end{aligned}$$

Exercise 8, Section 1.2.1 of Donald Knuth's *The Art of Programming, Volume 1* reads
(a) Prove the following theorem of Nicomachus (C.E. c.100) by induction:

$$1^3 = 1,\ 2^3 = 3+5,\ 3^3 = 7+9+11,\ 4^3 = 13+15+17+19,\ \text{etc.}$$

(b) Use this result to prove the remarkable formula

$$1^3 + 2^3 + \cdots + n^3 = (1+2+\cdots+n)^2\ .$$

Recall from Homework 2,

$$\vdash\ (+k \mid 1 \leq k \leq n : 2k-1) = n^2\ .$$

The following exercises provide a proof of (b) based on Knuth's approach. As in the previous problem, a change of dummy result lies at the core of the proof part (c).

4. (a) Use Mathematical Induction to prove that provided k d.n.o.f. in P,

$$\vdash\ (+k \mid 1 \leq k \leq n : P) \ = \ n \cdot P\ .$$

(b) Prove

$$\vdash\ (+j \mid 1 \leq j \leq n : j^3) \ = \ (+j \mid 1 \leq j \leq n : (+k \mid 1 \leq k \leq j : j^2 - j + (2k-1)))\ .$$

(c) Prove

$$\vdash\ (n+1)^2 - (n+1) = 2 \cdot (+k \mid 1 \leq k \leq n : k)$$

and use Mathematical Induction to prove

$$\vdash\ (+j \mid 1 \leq j \leq n : (+k \mid 1 \leq k \leq j : j^2 - j + (2k-1))) \ = \ (+k \mid 1 \leq k \leq (+j \mid 1 \leq j \leq n : j) : 2k-1)\ .$$

(d) Prove

$$\vdash (+j \mid 1 \leq j \leq n : j^3) \ = \ (+j \mid 1 \leq j \leq n : j)^2\ .$$

There are additional exercises requiring Mathematical Induction on the Chapter 14 Exercise List.