Chapter 14 Exercise List
SOLUTIONS

1. (a) i. Prove,

$$\vdash (\exists y \mid y \in T : z = \langle s, y\rangle) \;\Rightarrow\; (+y \mid z = \langle s, y\rangle : 1) = 1 \ .$$

Answer: By (9.19) and (9.30) it suffices to prove

$$w \in T \;\wedge z = \langle s, w\rangle \;\Rightarrow\; (+y \mid z = \langle s, y\rangle : 1) = 1 \ .$$

Assume $w \in T$ and assume $z = \langle s, w\rangle$. We are using $=$ for **type** $\mathbb{N}$ in the left margin.

$$\begin{array}{cl}
 & (+y \mid z = \langle s, y\rangle : 1) \\
= & \langle \text{ Assumption, Leibniz (8.12)(i), } y \text{ d.n.o.f } w \in T,\ z = \langle s, w\rangle \ \rangle \\
 & (+y \mid \langle s, w\rangle = \langle s, y\rangle : 1) \\
= & \langle \text{ (11.4) } \rangle \\
 & (+y \mid s = s \ \wedge\ w = y : 1) \\
= & \langle \text{ (1.2), Metatheorem (3.7) } \rangle \\
 & (+y \mid \mathit{true} \ \wedge\ w = y : 1) \\
= & \langle \text{ (3.39), (1.3) } \rangle \\
 & (+y \mid y = w : 1) \\
= & \langle \text{ (8.14), } y \text{ d.n.o.f. } w \ \rangle \\
 & 1 \ .
\end{array}$$

ii. Fill in reasons and complete the following proof that

$$\vdash \#(\{s\} \times T) \;=\; \#T \ .$$

Answer: Let $z,\ x,\ y,\ w$ be fresh variables. We are using $=$ for **type** $\mathbb{N}$ in the left margin.

$$\begin{array}{cl}
 & \#(\{s\} \times T) \\
= & \langle \text{ (11.12), } z \text{ d.n.o.f. } \{s\} \times T \ \rangle \\
 & (+z \mid z \in \{s\} \times T : 1) \\
= & \langle \text{ (14.3)}', \text{ (11.7), x, y d.n.o.f. } \{s\} \times T,\ 1 \ \rangle \\
 & (+z \mid (\exists x, y \mid x \in \{s\} \ \wedge\ y \in T : z = \langle x, y\rangle) : 1) \\
= & \langle \text{ Lemma, } \vdash x \in \{s\} \ \equiv x = s \ \rangle \\
 & (+z \mid (\exists x, y \mid x = s \ \wedge\ y \in T : z = \langle x, y\rangle) : 1) \\
= & \langle \text{ (8.20), } y \text{ d.n.o.f. } x = s \ \rangle \\
 & (+z \mid (\exists x \mid x = s : (\exists y \mid y \in T : z = \langle x, y\rangle)) : 1) \\
= & \langle \text{ (8.14), } x \text{ d.n.o.f. } s \ \rangle \\
 & (+z \mid (\exists y \mid y \in T : z = \langle s, y\rangle) : 1)
\end{array}$$

$$= \quad \left\langle \text{ Part i. } \right\rangle$$
$$(+z \,|\, (\exists y \,|\, y \in T : z = \langle s, y \rangle) : (+y \,|\, z = \langle s, y \rangle : 1))$$
$$= \quad \left\langle \text{ (8.21), } w \text{ d.n.o.f. } y \in T,\ z = \langle s, y \rangle \right\rangle$$
$$(+z \,|\, (\exists w \,|\, w \in T : z = \langle s, w \rangle) : (+y \,|\, z = \langle s, y \rangle : 1))$$
$$= \quad \left\langle \text{ (8.20), } y \text{ d.n.o.f. } (\exists w \,|\, w \in T : z = \langle s, w \rangle) \right\rangle$$
$$(+z, y \,|\, (\exists w \,|\, w \in T : z = \langle s, w \rangle) \wedge z = \langle s, y \rangle : 1)$$
$$= \quad \left\langle \text{ (9.21), } w \text{ d.n.o.f. } z = \langle s, y \rangle \right\rangle$$
$$(+z, y \,|\, (\exists w \,|\, w \in T : z = \langle s, w \rangle \wedge z = \langle s, y \rangle) : 1)$$
$$= \quad \left\langle \text{ (3.84)(a) } \right\rangle$$
$$(+z, y \,|\, (\exists w \,|\, w \in T : \langle s, y \rangle = \langle s, w \rangle \wedge z = \langle s, y \rangle) : 1)$$
$$= \quad \left\langle \text{ (11.4) } \right\rangle$$
$$(+z, y \,|\, (\exists w \,|\, w \in T : s = s \wedge y = w \wedge z = \langle s, y \rangle) : 1)$$
$$= \quad \left\langle \text{ (1.2), Metatheorem (3.7) } \right\rangle$$
$$(+z, y \,|\, (\exists w \,|\, w \in T : \mathit{true} \wedge y = w \wedge z = \langle s, y \rangle) : 1)$$
$$= \quad \left\langle \text{ (3.39), (9.19), (1.3) } \right\rangle$$
$$(+z, y \,|\, (\exists w \,|\, w = y : w \in T \wedge z = \langle s, y \rangle) : 1)$$
$$= \quad \left\langle \text{ (8.14), } w \text{ d.n.o.f. } y \right\rangle$$
$$(+z, y \,|\, y \in T \wedge z = \langle s, y \rangle : 1)$$
$$= \quad \left\langle \vdash (\star z, y | R : P) = (\star y, z | R : P) \right\rangle$$
$$(+y, z \,|\, y \in T \wedge z = \langle s, y \rangle : 1)$$
$$= \quad \left\langle \text{ (8.20), } z \text{ d.n.o.f. } y \in T \right\rangle$$
$$(+y \,|\, y \in T : (+z \,|\, z = \langle s, y \rangle : 1))$$
$$= \quad \left\langle \text{ (8.14), } z \text{ d.n.o.f. } \langle s, y \rangle \right\rangle$$
$$(+y \,|\, y \in T : 1)$$
$$= \quad \left\langle \text{ (11.12), } y \text{ d.n.o.f. } T \right\rangle$$
$$\#T \ .$$

(b) Prove,

$$\vdash (s \neq s') \;\Rightarrow\; (\,\#(\{s, s'\} \times T) \;=\; \#T + \#T\,) \ .$$

Answer: The proof follows easily from (8.16) and Part (a) once we prove two preliminary results,

$$\vdash \{s, s'\} \times T \;=\; (\{s\} \times T) \cup (\{s'\} \times T)$$

and

$$\vdash s \neq s' \;\Rightarrow\; (\{s\} \times T) \cap (\{s'\} \times T) \;=\; \emptyset \ .$$

Choose x, y to be fresh variables. By (9.16) and 14RE to prove the first result it suffices to prove

$$\vdash \langle x, y \rangle \in \{s, s'\} \times T \;\equiv\; \langle x, y \rangle \in (\{s\} \times T) \cup (\{s'\} \times T) \ .$$

$$\begin{array}{ll}
& \langle x,y\rangle \in \{s,s'\} \times T \\
= & \big\langle\ (14.4)\ \big\rangle \\
& x \in \{s,s'\}\ \wedge\ y \in T \\
= & \big\langle\ \text{Notation}\ \big\rangle \\
& x \in \{x \,|\, x = s\ \vee\ x = s'\}\ \wedge\ y \in T \\
= & \big\langle\ (11.7)\ \big\rangle \\
& (x = s\ \vee\ x = s')\ \wedge\ y \in T \\
= & \big\langle\ \text{Lemma, } \vdash\ x \in \{s\}\ \equiv\ x = s\ \big\rangle \\
& (x \in \{s\}\ \vee\ x \in \{s'\})\ \wedge\ y \in T \\
= & \big\langle\ (3.46)\ \big\rangle \\
& (x \in \{s\}\ \wedge\ y \in T)\ \vee\ (x \in \{s'\}\ \wedge\ y \in T) \\
= & \big\langle\ (14.4)\ \big\rangle \\
& \langle x,y\rangle \in (\{s\} \times T)\ \vee\ \langle x,y\rangle \in (\{s'\} \times T) \\
= & \big\langle\ (11.20)\ \big\rangle \\
& \langle x,y\rangle \in (\{s\} \times T)\ \cup\ (\{s'\} \times T)\ .
\end{array}$$

Similarly, for the second, assume $s \neq s'$ and choose $x,\ y$ to be fresh variables. By 14RP it suffices to prove

$$\vdash \langle x,y\rangle \in (\{s\} \times T)\ \cap\ (\{s'\} \times T)\ \equiv\ \mathit{false}\ .$$

$$\begin{array}{ll}
& \langle x,y\rangle \in (\{s\} \times T)\ \cap\ (\{s'\} \times T) \\
= & \big\langle\ (11.21)\ \big\rangle \\
& \langle x,y\rangle \in (\{s\} \times T)\ \wedge\ \langle x,y\rangle \in (\{s'\} \times T) \\
= & \big\langle\ (14.4)\ \big\rangle \\
& x \in \{s\}\ \wedge y \in T\ \ \wedge\ \ x \in \{s'\}\ \wedge y \in T \\
= & \big\langle\ \text{Lemma, } \vdash\ x \in \{s\}\ \equiv\ x = s, \text{(3.36), (3.38)}\ \big\rangle \\
& x = s\ \wedge\ x = s'\ \wedge\ y \in T \\
= & \big\langle\ \text{(3.84)(a)}\ \big\rangle \\
& x = s\ \wedge\ s = s'\ \wedge\ y \in T \\
= & \big\langle\ \text{Assumption}\ \big\rangle \\
& x = s\ \wedge\ \mathit{false}\ \wedge\ y \in T \\
= & \big\langle\ (3.40)\ \big\rangle \\
& \mathit{false}\ .
\end{array}$$

Now assume $s \neq s'$. Choose z to be fresh. We are using $=$ for **type** $\mathbb{N}$ in the left margin.

$$\begin{array}{ll}
& \#(\{s,s'\} \times T) \\
= & \big\langle\ (11.12)\ \big\rangle
\end{array}$$

$$\begin{array}{cl}
& (+z \,|\, z \in (\{s, s'\} \times T) : 1) \\
= & \langle \text{ First preliminary result, (11.20) } \rangle \\
& (+z \,|\, z \in (\{s\} \times T) \;\vee\; z \in (\{s'\} \times T) : 1) \\
= & \langle \text{ (8.16), using second preliminary result } \rangle \\
& (+z \,|\, z \in (\{s\} \times T) : 1) \;+\; (+z \,|\, z \in (\{s'\} \times T) : 1) \\
= & \langle \text{ Part (a) } \rangle \\
& \#T + \#T \;.
\end{array}$$

The solution can be extended to provide a proof of (14.15) by Mathematical Induction. This avoids the need to introduce Axiom (15.51).

2 Prove,

$$\vdash \text{Dom.}(\rho \circ \sigma) \subseteq \text{Dom.}\rho \;.$$

Answer: Take x, y, z, w to be fresh variables.

$$\begin{array}{cl}
& \text{Dom.}(\rho \circ \sigma) \subseteq \text{Dom.}\rho \\
= & \langle \text{ (11.13) } \rangle \\
& (\forall z \,|\, z \in \text{Dom.}(\rho \circ \sigma) : z \in \text{Dom.}\rho \\
= & \langle \text{ (14.16), (11.7) } \rangle \\
& (\forall z \,|\, (\exists y \,|\, : \langle z, y\rangle \in \rho \circ \sigma) : (\exists x \,|\, \langle z, x\rangle \in \rho)) \\
= & \langle \text{ (9.2) } \rangle \\
& (\forall z \,|\, : (\exists y \,|\, : \langle z, y\rangle \in \rho \circ \sigma) \;\Rightarrow\; (\exists x \,|\, \langle z, x\rangle \in \rho))
\end{array}$$

By (9.16) it suffices to prove

$$(\exists y \,|\, : \langle z, y\rangle \in \rho \circ \sigma) \;\Rightarrow\; (\exists x \,|\, \langle z, x\rangle \in \rho)$$

and by (9.30) it suffices to prove

$$\langle z, w\rangle \in \rho \circ \sigma \;\Rightarrow\; (\exists x \,|\, \langle z, x\rangle \in \rho) \;.$$

$$\begin{array}{cl}
& \langle z, w\rangle \in rho \circ \sigma \\
= & \langle \text{ (14.20) } \rangle \\
& (\exists x \,|\, : \langle z, x\rangle \in \rho \;\wedge\; \langle x, w\rangle \in \sigma) \\
\Rightarrow & \langle \text{ (9.26)}' \rangle
\end{array}$$

3 Prove, provided x, y do not have free occurrences in S or T,

$$\vdash S \neq \emptyset \wedge T \neq \emptyset \Rightarrow (\exists x, y \;\,|: \langle x, y\rangle \in S \times T) \;.$$

Answer: Take x, y to be fresh variables.

$$\begin{array}{cl}
& S \neq \emptyset \;\wedge\; T \neq \emptyset \\
= & \langle \text{ Definition of } \emptyset \text{ } \rangle
\end{array}$$

$$\begin{aligned}
& (\exists x \mid \ : x \in S) \ \wedge \ (\exists y \mid \ : y \in T) \\
= \ & \langle \ (9.21) \ \rangle \\
& (\exists x \mid \ : x \in S \ \wedge \ (\exists y \mid \ : y \in T)) \\
= \ & \langle \ (9.21) \ \rangle \\
& (\exists x \mid \ : (\exists y \mid \ : x \in S \ \wedge \ y \in T)) \\
= \ & \langle \ (8.20) \ \rangle \\
& (\exists x, y \mid \ : x \in S \ \wedge \ y \in T) \\
= \ & \langle \ (14.4) \ \rangle \\
& (\exists x, y \mid \ : \langle x, y \rangle \in S \times T)
\end{aligned}$$

4 Prove,

$$\vdash (S \times T \ \cap \ T \times S) = \emptyset \ \ \Rightarrow \ \ T \cap S = \emptyset \ .$$

Answer: Take z, w, x, y to be fresh variables.
By Contrapositive it suffices to prove $\vdash \ S \cap T \neq \emptyset \ \ \Rightarrow \ \ (S \times T \ \cap \ T \times S) \neq \emptyset$. Now,

$$\begin{aligned}
& S \cap T \neq \emptyset \ \ \Rightarrow \ \ (S \times T \ \cap \ T \times S) \neq \emptyset \\
= \ & \langle \ \text{Definition of } \emptyset \ \rangle \\
& (\exists z \mid \ : z \in S \cap T) \ \ \Rightarrow \ \ (S \times T \ \cap \ T \times S) \neq \emptyset
\end{aligned}$$

so that by (9.30) it suffices to prove $\vdash \ w \in S \cap T \ \ \Rightarrow \ \ (S \times T \ \cap \ T \times S) \neq \emptyset$.

$$\begin{aligned}
& w \in S \cap T \\
= \ & \langle \ (14.4) \ \rangle \\
& w \in S \wedge w \in T \\
= \ & \langle \ (3.38) \ \rangle \\
& w \in S \ \wedge \ w \in T \ \ \wedge \ \ w \in T \ \wedge \ w \in S \\
= \ & \langle \ (14.4) \ \rangle \\
& \langle w, w \rangle \in S \times T \ \wedge \ \langle w, w \rangle \in T \times S \\
= \ & \langle \ (11.21) \ \rangle \\
& \langle w, w \rangle \in S \times T \ \cap \ T \times S \\
\Rightarrow \ & \langle \ (9.28) \text{ for multiple quantifications} \ \rangle \\
& (\exists x, y \mid \langle x, y \rangle \in \mathbb{U} \times \mathbb{U} : \langle x, y \rangle \in S \times T \ \cap \ T \times S) \\
= \ & \left\langle \begin{array}{l} (9.17) \text{ for multiple quantifications, 14RP, } (9.17) \\ x, y \text{ d.n.o.f. } \mathbb{U} \times \mathbb{U}, z \in S \times T \ \cap \ T \times S, \ z \text{ d.n.o.f. } \mathbb{U} \times \mathbb{U} \end{array} \right\rangle \\
& (\exists z \mid z \in \mathbb{U} \times \mathbb{U} : z \in S \times T \ \cap \ T \times S) \\
\Rightarrow \ & \langle \ \text{Lemma: } \vdash \ (\exists z \mid R : P) \Rightarrow (\exists z \mid \ : P) \ \rangle \\
& (\exists z \mid \ : z \in S \times T \ \cap \ T \times S)
\end{aligned}$$

5 Prove,

$$\vdash S \neq \emptyset \ \wedge\ T \neq \emptyset \quad \equiv \quad S \times T \neq \emptyset \ .$$

Answer: Take $x,\ y,\ z$ to be fresh variables.

$$\begin{array}{ll}
 & S \neq \emptyset \ \wedge\ T \neq \emptyset \\
= & \big\langle \text{ Property of } \emptyset \ \big\rangle \\
 & (\exists x \,|\, x \in \mathbf{U} : x \in S) \ \wedge\ (\exists y \,|\, y \in \mathbf{U} : y \in T) \\
= & \big\langle \text{ (9.21)m } x \text{ d.n.o.f. in } (\exists y \,|\, x \in \mathbf{U} : y \in T) \ \big\rangle \\
 & (\exists x \,|\, x \in \mathbf{U} : x \in S \ \wedge\ (\exists y \,|\, y \in \mathbf{U} : y \in T)) \\
= & \big\langle \text{ (9.221), } y \text{ d.n.o.f. in } x \in S \ \big\rangle \\
 & (\exists x \,|\, x \in \mathbf{U} : (\exists y \,|\, y \in \mathbf{U} : x \in S \ \wedge\ y \in T)) \\
= & \big\langle \text{ (8.20), } y \text{ d.n.o.f. in } x \in \mathbf{U} \ \big\rangle \\
 & (\exists x, y \,|\, x \in \mathbf{U} \ \wedge\ y \in \mathbf{U} : x \in S \ \wedge\ y \in T) \\
= & \big\langle \text{ (14/4)}' \ \big\rangle \\
 & (\exists x, y \,|\, \langle x, y\rangle \in \mathbf{U} \times \mathbf{U} : \langle x, y\rangle \in S \times T) \\
= & \big\langle \text{ (9.17) } \big\rangle \\
 & \neg(\forall x, y \,|\, \langle x, y\rangle \in \mathbf{U} \times \mathbf{U} : \langle x, y\rangle \notin S \times T) \\
= & \big\langle \text{ 14 RP, } x,\ y \text{ d.n.o.f. in } \mathbf{U} \times \mathbf{U},\ z \notin S \times T,\ z \text{ d.n.o.f. in } \mathbf{U} \times \mathbf{U} \ \big\rangle \\
 & \neg(\forall z \,|\, z \in \mathbf{U} \times \mathbf{U} : z \notin S \times T) \\
= & \big\langle \text{ (9.17) } \big\rangle \\
 & (\exists z \,|\, z \in \mathbf{U} \times \mathbf{U} : z \in S \times T) \\
= & \big\langle \text{ (9.19) } \big\rangle \\
 & (\exists z \,|\ : z \in \mathbf{U} \times \mathbf{U} \ \wedge\ z \in S \times T) \\
= & \big\langle \text{ Lemma, } \vdash z \in S \times T \ \Rightarrow\ z \in \mathbf{U} \times \mathbf{U}\text{, (3.60) } \big\rangle \\
 & (\exists z \,|\ : z \in S \times T) \\
= & \big\langle \text{ Property of } \emptyset \ \big\rangle \\
 & S \times T \neq \emptyset \ .
\end{array}$$

6 Prove that

$$\vdash \ S \times T \ \subseteq\ S \times U \quad \Rightarrow \quad T \subseteq U \ \vee\ S = \emptyset \ .$$

Answer: There are simpler proofs than the one I have chosen to give below.

$$\begin{array}{ll}
 & S \times T \ \subseteq\ S \times U \quad \Rightarrow \quad T \subseteq U \ \vee\ S = \emptyset \\
= & \big\langle \text{ Lemma: } \vdash P \Rightarrow Q \ \equiv\ P \wedge \neg Q \ \Rightarrow\ \mathit{false} \ \big\rangle \\
 & S \times T \ \subseteq\ S \times U \ \wedge\ T \not\subseteq U \ \wedge\ S \neq \emptyset \quad \Rightarrow \quad \mathit{false} \ ,
\end{array}$$

it suffices to prove

$$\vdash S \times T \ \subseteq\ S \times U \ \wedge\ T \not\subseteq U \ \wedge\ S \neq \emptyset \quad \Rightarrow \quad \mathit{false} \ .$$

Choose x, y, z to be fresh variables.

$$
\begin{array}{ll}
 & S \times T \ \subseteq\ S \times U \ \wedge\ T \not\subseteq U \ \wedge\ S \neq \emptyset \\
= & \langle \text{ (11.13), Definition of } \emptyset. \ \rangle \\
 & (\forall z \,|\, z \in S \times T : z \in S \times U) \ \wedge\ \neg(\forall y \,|\, y \in T : y \in U) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
= & \langle \text{ 14 RP } \rangle \\
 & (\forall x, y \,|\, \langle x, y\rangle \in S \times T : \langle x, y\rangle \in S \times U) \ \wedge\ \neg(\forall y \,|\, y \in T : y \notin U) \ \wedge\ (\exists x \,|\, x \notin S) \\
= & \langle \text{ (14.4) } \rangle \\
 & (\forall x, y \,|\, x \in S \ \wedge\ y \in T : x \in S \ \wedge\ y \in U) \ \wedge\ (\exists y \,|\, y \in T : y \notin U) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
= & \langle \text{ (8.20) } \rangle \\
 & (\forall x \,|\, x \in S :\ (\forall y \,|\, y \in T : x \in S \ \wedge\ y \in U)) \ \wedge\ (\exists y \,|\, y \in T : y \notin U) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
= & \langle \text{ (9.7), (3.62) } \rangle \\
 & (\forall x \,|\, x \in S : (\exists y \,|\, y \in T : y \notin U) \ \wedge\ (\forall y \,|\, y \in T : x \in S \ \wedge\ y \in U)) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
= & \langle \text{ (9.21) } \rangle \\
 & (\forall x \,|\, x \in S : (\exists y \,|\, y \in T : y \notin U \ \wedge\ (\forall y \,|\, y \in T : x \in S \ \wedge\ y \in U))) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
= & \langle \text{ (9.19) } \rangle \\
 & (\forall x \,|\, x \in S : (\exists y \,|\ : y \in T \ \wedge\ y \notin U \ \wedge\ (\forall y \,|\, y \in T : x \in S \ \wedge\ y \in U))) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
= & \langle \text{ (9.2) } \rangle \\
 & (\forall x \,|\, x \in S : (\exists y \,|\ : y \in T \ \wedge\ y \notin U \ \wedge\ (\forall y \,|\ : y \in T \Rightarrow x \in S \ \wedge\ y \in U))) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
\Rightarrow & \langle \text{ (4.3), (9.27), Modus Ponens } \rangle \\
 & (\forall x \,|\, x \in S : (\exists y \,|\ : y \in T \ \wedge\ y \notin U \ \wedge\ (y \in T \Rightarrow x \in S \ \wedge\ y \in U))) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
= & \langle \text{ (3.66) } \rangle \\
 & (\forall x \,|\, x \in S : (\exists y \,|\ : y \in T \ \wedge\ y \notin U \ \wedge\ x \in S \ \wedge\ y \in U)) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
= & \langle \text{ (3.42) } \rangle \\
 & (\forall x \,|\, x \in S : (\exists y \,|\ : \mathit{false})) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
= & \langle \text{ (9.24) } \rangle \\
 & (\forall x \,|\, x \in S : \mathit{false}) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
= & \langle \text{ (9.2) } \rangle \\
 & (\forall x \,|\ : x \in S \ \Rightarrow\ \mathit{false}) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
= & \langle \text{ (3.74) } \rangle \\
 & (\forall x \,|\ : x \notin S) \ \wedge\ \neg(\forall x \,|\, x \notin S) \\
= & \langle \text{ (3.42) } \rangle \\
 & \mathit{false}
\end{array}
$$

7 (a) Prove,

$$\vdash\ \langle x, y\rangle \in \imath_B \ \equiv\ x \in B \ \wedge\ x = y\ .$$

Answer:

$$\langle x, y\rangle \in \imath_B$$

$$
\begin{array}{ll}
= & \big\langle \text{ Definition of } \imath_B \ \big\rangle \\
 & \langle x, y\rangle \in \{z \mid z \in B : \langle z, z\rangle\} \\
= & \big\langle \ (11.3) \ \big\rangle \\
 & (\exists z \mid z \in B : \langle x, y\rangle = \langle z, z\rangle) \\
= & \big\langle \ (14.4) \ \big\rangle \\
 & (\exists z \mid z \in B : x = z \ \wedge \ y = z) \\
= & \big\langle \ (9.19) \ \big\rangle \\
 & (\exists z \mid z = x : z \in B \ \wedge \ y = z) \\
= & \big\langle \ (8.14) \ \big\rangle \\
 & x \in B \ \wedge \ y = z
\end{array}
$$

(b) Prove,

$$\vdash \imath_B^{-1} = \imath_B \ .$$

Answer: By 14 RE and (9.16) for multiples, it suffices to prove

$$\langle x, y\rangle \in \imath_B^{-1} \ \equiv \ \langle x, y\rangle \in \imath_B \ .$$

$$
\begin{array}{ll}
 & \langle x, y\rangle \in \imath_B^{-1} \\
= & \big\langle \ (14.18) \ \big\rangle \\
 & \langle y, x\rangle \in \imath_B \\
= & \big\langle \text{ Part (a) } \big\rangle \\
 & y \in B \ \wedge \ x = y \\
= & \big\langle \ (3.84)(\text{b}) \ \big\rangle \\
 & x \in B \ \wedge \ x = y \\
= & \big\langle \ (1.3) \ \big\rangle \\
 & x \in B \ \wedge \ y = x \\
= & \big\langle \text{ part (a) } \big\rangle \\
 & \langle x, y\rangle \in \imath_B
\end{array}
$$

(c) Let ρ be a relation. Prove,

$$\vdash \rho \subseteq B \times B \ \Rightarrow \ \imath_B \circ \rho = \rho \ .$$

Answer: Take x, y, z to be fresh variables.

$$
\begin{array}{ll}
 & \rho \subseteq B \times B \ \Rightarrow \ \imath_B \circ \rho = \rho \\
= & \big\langle \text{ 14 RE } \big\rangle \\
 & \rho \subseteq B \times B \ \Rightarrow \ (\forall x, y \mid \ : \langle x, y\rangle \in \imath_B \circ \rho \ \equiv \ \langle x, y\rangle \in \rho) \\
= & \big\langle \text{ Lemma: } \vdash P \Rightarrow (\forall x, y \mid \ : Q) \ \equiv \ (\forall x \mid \ : P \Rightarrow Q) \ \big\rangle \\
 & (\forall x, y \mid \ : \rho \subseteq B \times B \ \Rightarrow \ (\langle x, y\rangle \in \imath_B \circ \rho \ \equiv \ \langle x, y\rangle \in \rho))
\end{array}
$$

By (9.16) for multiples it suffices to prove

$$\rho \subseteq B \times B \;\Rightarrow\; (\langle x,y\rangle \in \imath_B \circ \rho \;\equiv\; \langle x,y\rangle \in \rho)$$

and by (3.62) this is syntactically equivalent to

$$\rho \subseteq B \times B \,\wedge\, \langle x,y\rangle \in \imath_B \circ \rho \;\equiv\; \rho \subseteq B \times B \,\wedge\, \langle x,y\rangle \in \rho\ .$$

$$\begin{array}{ll}
 & \rho \subseteq B \times B \,\wedge\, \langle x,y\rangle \in \imath_B \circ \rho \\
= & \big\langle\ (14.20)\ \big\rangle \\
 & \rho \subseteq B \times B \,\wedge\, (\exists z \,|\, : \langle x,z\rangle \in \imath_B \,\wedge\, \langle z,y\rangle \in \rho) \\
= & \big\langle\ \text{Part (a)}\ \big\rangle \\
 & \rho \subseteq B \times B \,\wedge\, (\exists z \,|\, : x \in B \,\wedge\, z = x \,\wedge\, \langle z,y\rangle \in \rho) \\
= & \big\langle\ (9.19)\ \big\rangle \\
 & \rho \subseteq B \times B \,\wedge\, (\exists z \,|\, z = x : x \in B \,\wedge\, \langle z,y\rangle \in \rho) \\
= & \big\langle\ (8.14)\ \big\rangle \\
 & \rho \subseteq B \times B \,\wedge\, x \in B \,\wedge\, \langle x,y\rangle \in \rho \\
= & \big\langle\ \text{Lemma: } \vdash \rho \subseteq B \times B \,\wedge\, \langle x,y\rangle \in \rho \;\Rightarrow\; x \in B,\ (3.60)\ \big\rangle \\
 & \rho \subseteq B \times B \,\wedge\, \langle x,y\rangle \in \rho
\end{array}$$

8 (a) Prove that provided z d.n.o.f. Q, $\vdash (\forall z \,|\, P : Q) \;\equiv\; ((\exists z \,|\, : P) \Rightarrow Q$.
Answer:

$$\begin{array}{ll}
 & (\forall z \,|\, P : Q) \\
= & \big\langle\ (9.2)\ \big\rangle \\
 & (\forall z \,|\, : P \Rightarrow Q) \\
= & \big\langle\ (3.59)\ \big\rangle \\
 & (\forall z \,|\, : \neg P \vee Q) \\
= & \big\langle\ (9.5),\ z \text{ d.n.o.f. } Q\ \big\rangle \\
 & (\forall z \,|\, : \neg P) \,\vee\, Q \\
= & \big\langle\ (9.18)\text{(a)}\ \big\rangle \\
 & \neg(\exists z \,|\, : P) \,\vee\, Q \\
= & \big\langle\ (3.59)\ \big\rangle \\
 & (\exists z \,|\, : P) \;\Rightarrow\; Q
\end{array}$$

(b) Prove, $\vdash \rho \subseteq B \times C \;\Rightarrow\; Dom.\rho \subseteq B$.
Answer: Take z, x, y to be fresh.

$$\begin{array}{ll}
 & \rho \subseteq B \times C \\
= & \big\langle\ (11.13)\ \big\rangle \\
 & (\forall z \,|\, z \in \rho : z \in B \times C
\end{array}$$

$$
\begin{aligned}
= &\quad \langle\ \text{14RP, } x, y \text{ d.n.o.f. } \rho, z \in B \times C, \ z \text{ d.n.o.f. } \rho\ \rangle \\
&\quad (\forall x, y \mid \langle x, y\rangle \in \rho : \langle x, y\rangle \in B \times C \\
= &\quad \langle\ (14.4)'\ \rangle \\
&\quad (\forall x, y \mid \langle x, y\rangle \in \rho : x \in B \ \wedge \ y \in C))\rangle \in B \times C \\
\Rightarrow &\quad \langle\ (9.11) \text{ for multiple quantifications }\ \rangle \\
&\quad (\forall x, y \mid \langle x, y\rangle \in \rho : x \in B) \\
= &\quad \langle\ (8.20), \ y \text{ d.n.o.f } \mathit{true}\ \rangle \\
&\quad (\forall x \mid \ : (\forall y \mid \langle x, y\rangle \in \rho : x \in B)) \\
= &\quad \langle\ \text{Part (a), } y \text{ d.n.o.f. } x \in B\ \rangle \\
&\quad (\forall x \mid \ : (\exists y \mid \langle x, y\rangle \in \rho) \Rightarrow x \in B) \\
= &\quad \langle\ (9.2)\ \rangle \\
&\quad (\forall x \mid (\exists y \mid \langle x, y\rangle \in \rho) : x \in B) \\
= &\quad \langle\ (11.7)\ \rangle \\
&\quad (\forall x \mid x \in \{x \mid (\exists y \mid \langle x, y\rangle \in \rho)\} : x \in B) \\
= &\quad \langle\ \text{Definition of } \mathit{Dom}.\rho\ \rangle \\
&\quad (\forall x \mid x \in \mathit{Dom}.\rho : x \in B) \\
= &\quad \langle\ (11.13)\ \rangle \\
&\quad \mathit{Dom}.\rho \subseteq B \ .
\end{aligned}
$$

9 Let ρ be a relation, and suppose that n does not occur freely in ρ .

Use Mathematical Induction to prove that $\vdash \ (\forall n : \mathbb{N} \mid 1 \leq n : \ \text{Dom.}\, \rho^n \subseteq \text{Dom.}\, \rho)$.

Answer:

The Base Case is $n = 1$.

$$
\begin{aligned}
&\quad \text{Dom.}\rho^1 \subseteq \text{Dom.}\rho \\
= &\quad \langle\ (14.25)\ \rangle \\
&\quad \text{Dom.}\rho \subseteq \text{Dom.}\rho
\end{aligned}
$$

For the induction step, assume Dom.$\rho^n \subseteq$ Dom.ρ. Recall that in Problem 2 we proved, $\vdash$ Dom.$(\rho \circ \sigma) \subseteq$ Dom.ρ so the first line of the following is a theorem

$$
\begin{aligned}
&\quad \text{Dom.}(\rho^n \circ \rho) \subseteq \text{Dom.}\rho^n \\
= &\quad \langle\ \text{Assumption, (3.39)}\ \rangle \\
&\quad \text{Dom.}(\rho^n \circ \rho) \subseteq \text{Dom.}\rho^n \ \ \wedge \ \ \text{Dom.}\rho^n \subseteq \text{Dom.}\rho \\
\Rightarrow &\quad \langle\ (11.59)\ \rangle \\
&\quad \text{Dom.}(\rho^n \circ \rho) \subseteq \text{Dom.}\rho \\
= &\quad \langle\ (14.25)\ \rangle \\
&\quad \text{Dom.}\rho^{n+1} \subseteq \text{Dom.}\rho
\end{aligned}
$$

By Modus Ponens, we have that Dom.$\rho^{n+1} \subseteq$ Dom.ρ is a temporary theorem.

10 Let ρ be a relation on S which is reflexive on S and transitive. Prove that $\rho = \rho^2$.
Answer: Assume

$$\rho \subseteq S \times S \ \wedge\ \rho \text{ reflexivie on } S \ \wedge\ \rho \text{ transitive.}$$

We will prove $\rho = \rho^2$ by Mutual Implication.

For $\rho \subseteq \rho^2$, take x, y, z to be fresh variables.

$$\begin{array}{ll} & \rho \subseteq \rho^2 \\ = & \langle\ (11.13)\ \rangle \\ & (\forall z \mid z \in \rho : z \in \rho^2) \\ = & \langle\ 14 \text{ RP}\ \rangle \\ & (\forall x, y \mid \langle x, y\rangle \in \rho : \langle x, y\rangle \in \rho^2) \\ = & \langle\ (9.2)\ \rangle \\ & (\forall x, y \mid\ : \langle x, y\rangle \in \rho \ \Rightarrow\ \langle x, y\rangle \in \rho^2) \end{array}$$

By (9.16) for multiples it is enough to prove $\langle x, y\rangle \in \rho \ \Rightarrow\ \langle x, y\rangle \in \rho^2$.

$$\begin{array}{ll} & \langle x, y\rangle \in \rho \\ = & \langle\ \text{Assumption gives } \rho \subseteq S \times S\ \rangle \\ & \rho \subseteq S \times S \ \wedge\ \langle x, y\rangle \in \rho \\ = & \langle\ \text{Lemma: } \vdash \rho \subseteq S \times S \ \wedge\ \langle x, y\rangle \in \rho \ \Rightarrow\ x \in S, \text{ (3.60)}\ \rangle \\ & \rho \subseteq S \times S \ \wedge\ \langle x, y\rangle \in \rho \ \wedge\ x \in S \\ = & \langle\ \text{Temporary Theorem: } x \in S \ \Rightarrow\ \langle x, x\rangle \in \rho, \text{ (3.60)}\ \rangle \\ & \rho \subseteq S \times S \ \wedge\ \langle x, y\rangle \in \rho \ \wedge\ x \in S \ \wedge\ \langle x, x\rangle \in \rho \\ \Rightarrow & \langle\ \text{(3.76)(b)}\ \rangle \\ & \langle x, x\rangle \in \rho \ \wedge\ \langle x, y\rangle \in \rho \\ = & \langle\ (14.25)\ \rangle \\ & \langle x, y\rangle \in \rho^2 \end{array}$$

For $\rho^2 \subseteq \rho$, take x, y, z, w.
The identical argument as above shows that it suffices to prove $\langle x, y\rangle \in \rho^2 \ \Rightarrow\ \langle x, y\rangle \in \rho$.

$$\begin{array}{ll} & \langle x, y\rangle \in \rho^2 \ \Rightarrow\ \langle x, y\rangle \in \rho \\ = & \langle\ (14.20)\ \rangle \\ & (\exists z \mid \langle x, z\rangle \in \rho \ \wedge\ \langle z, y\rangle \in \rho) \ \Rightarrow\ \langle x, y\rangle \in \rho \end{array}$$

We may apply Strong (9.30) as z does not occur free in the Assumption. It suffices to prove

$$\langle x, w\rangle \in \rho \ \wedge\ \langle w, y\rangle \in \rho \ \Rightarrow\ \langle x, y\rangle \in \rho\ .$$

This follows from the temporary theorem, ρ is transitive, by (9.13) for multiples and (Strong) Modus Ponens.

11 Assume that ρ is a binary relation on a non-empty set S. Decide which of the following are theorems. Prove or justify your answers.

(a) $r(\rho)$ is symmetric.
Answer: $r(\rho)$ is not necessarily symmetric. Take S to be $\{0,1\}$.
If ρ is $\{\langle 0,1\rangle\}$, then $r(\rho)$ is $\{\langle 0,0\rangle,\ \langle 1,1\rangle,\ \langle 0,1\rangle\}$ which is not symmetric.

(b) ρ^2 is transitive.
Answer: ρ^2 is not necessarily transitive. Take S to be $\{0,1,2,3,4\}$.
If ρ is $\{\langle 0,1\rangle,\ \langle 1,2\rangle,\ \langle 2,3\rangle,\ \langle 3,4\rangle\}$ then ρ^2 is $\{\langle 0,2\rangle,\ \langle 1,3\rangle,\ \langle 2,4\rangle\}$.
ρ^2 is not transitive as $\langle 0,2\rangle$ and $\langle 2,4\rangle$ are in ρ^2 but $\langle 0,4\rangle$ is not.

(c) $s(r(\rho)) = r(s(\rho))$.
Answer: We have $\vdash\ s(r(\rho)) = r(s(\rho))$.

$$\begin{array}{ll}
 & s(r(\rho)) = r(s(\rho)) \\
= & \langle\ (14.32)(\text{b}),\ (14.32)(\text{a})\ \rangle \\
 & r(\rho)\ \cup\ (r(\rho))^{-1}\ =\ s(\rho)\ \cup\ \imath_S \\
= & \langle\ (14.32)(\text{a}),\ (14.32)(\text{b})\ \rangle \\
 & (\rho \cup \imath_S)\ \cup\ (\rho \cup \imath_S)^{-1}\ =\ \rho\ \cup\ \rho^{-1}\ \cup\ \imath_S \\
= & \left\langle \begin{array}{l} \text{Lemma}, \vdash\ (\rho \cup \tau)^{-1}\ =\ \rho^{-1} \cup \tau^{-1} \\ \text{Lemma}, \vdash\ \imath_S\ =\ \imath_S^{-1} \end{array} \right\rangle \\
 & (\rho \cup \imath_S)\ \cup\ (\rho^{-1} \cup \imath_S)\ =\ \rho\ \cup\ \rho^{-1}\ \cup\ \imath_S \\
= & \langle\ (11.27),\ (11.26),\ (11.28)\ \rangle \\
 & \rho\ \cup\ \rho^{-1}\ \cup\ \imath_S\ =\ \rho\ \cup\ \rho^{-1}\ \cup\ \imath_S\ .
\end{array}$$

Choose $x,\ y$ fresh variables. To prove $\vdash\ (\rho \cup \tau)^{-1}\ =\ \rho^{-1} \cup \tau^{-1}$, it suffices by (9.16) and 14RE to prove

$$\vdash \langle x,y\rangle \in (\rho\ \cup\ \tau)^{-1}\ \equiv\ \langle x,y\rangle \in \rho^{-1}\ \cup\ \tau^{-1}\ .$$

$$\begin{array}{ll}
 & \langle x,y\rangle \in (\rho\ \cup\ \tau)^{-1} \\
= & \langle\ (14.18)\ \rangle \\
 & \langle y,x\rangle \in (\rho\ \cup\ \tau)^{-1} \\
= & \langle\ (11.20)\ \rangle \\
 & \langle y,x\rangle \in \rho\ \cup\ \langle y,x\rangle \in \tau \\
= & \langle\ (14.18)\ \rangle \\
 & \langle x,y\rangle \in \rho^{-1}\ \cup\ \langle x,y\rangle \in \tau^{-1} \\
= & \langle\ (11.20)\ \rangle \\
 & \langle x,y\rangle \in \rho^{-1}\ \cup\ \tau^{-1}
\end{array}$$

Choose $x,\ y,\ z$ fresh variables. To prove $\vdash\ \imath_S\ =\ \imath_S^{-1}$, it suffices by (9.16) and 14RE to prove

$$\vdash\ \langle x,y\rangle \in \imath_S\ \equiv\ \langle x,y\rangle \in \imath_S^{-1}\ .$$

$$
\begin{array}{ll}
 & \langle x, y\rangle \in \imath_S \\
= & \big\langle \text{ Definition of } \imath_S \ \big\rangle \\
 & \langle x, y\rangle \in \{z \mid z \in S : \langle z, z\rangle\} \\
= & \big\langle \text{ (11.3), } z \text{ d.n.o.f. } \langle x, y\rangle \ \big\rangle \\
 & (\exists z \mid z \in S : \langle x, y\rangle = \langle z, z\rangle) \\
= & \big\langle \text{ (14.4) } \big\rangle \\
 & (\exists z \mid z \in S : x = z \ \wedge \ y = z) \\
= & \big\langle \text{ (3.36) } \big\rangle \\
 & (\exists z \mid z \in S : y = z \ \wedge \ x = z) \\
= & \big\langle \text{ (14.4) } \big\rangle \\
 & (\exists z \mid z \in S : \langle y, x\rangle = \langle z, z\rangle) \\
= & \big\langle \text{ (11.3), } z \text{ d.n.o.f. } \langle y, x\rangle \ \big\rangle \\
 & \langle y, x\rangle \in \{z \mid z \in S : \langle z, z\rangle\} \\
= & \big\langle \text{ Definition of } \imath_S \ \big\rangle \\
 & \langle y, x\rangle \in \imath_S \\
= & \big\langle \text{ (14.18) } \big\rangle \\
 & \langle x, y\rangle \in \imath_S^{-1} \ .
\end{array}
$$

(d) $(s(\rho))^+$ is reflexive on S.
Answer: It is not necessarily the case that $(s(\rho))^+$ is reflexive on S.
Take S to be $\{0, 1, 2\}$. If ρ is $\{\langle 0, 1\rangle\}$ then $s(\rho)$ is $\{\langle 0, 1\rangle,\ \{\langle 1, 0\rangle\}$ and $(s(\rho))^+$ is $\{\langle 0, 0\rangle,\ \langle 1, 1\rangle,\ \langle 0, 1\rangle,\ \langle 1, 0\rangle\}$. $(s(\rho))^+$ is not reflexive on S as $\langle 2, 2\rangle \notin S$.

12 Recall that for x not occurring free in S, $\imath_S = \{x \mid x \in S : \langle x, x\rangle\}$.

(a) Prove,

$$\vdash \ \langle x, y\rangle \in \imath_S \ \equiv \ x \in S \ \wedge \ y = x \ .$$

Answer: This is identical to Part (a) of Problem 7.

(b) Prove,

$$\vdash \ \imath_S \text{ is an equivalence relation on } S \ .$$

Answer: We must show that $\imath_S$ is reflexive on S, symmetric, and transitive.

For reflexivity, take x fresh. By (9.16) it suffices to prove $x \in S \ \Rightarrow \ \langle x, x\rangle \in \imath_S$.

$$
\begin{array}{ll}
 & x \in S \\
= & \big\langle \text{ (1.2), (3.39) } \big\rangle \\
 & x \in S \ \wedge \ x = x \\
= & \big\langle \text{ Part (a) } \big\rangle \\
 & \langle x, x\rangle \in \imath_S \\
\Rightarrow & \big\langle \text{ (3.71) } \big\rangle \\
 & \langle x, x\rangle \in \imath_S
\end{array}
$$

For symmetry, take x, y to be fresh. By (9.16) it suffices to prove $\langle x,y\rangle \in \imath_S \;\equiv\; \langle y,x\rangle \in \imath_S$

$$
\begin{array}{cl}
 & \langle x,y\rangle \in \imath_S \\
= & \langle\ \text{Part (a)}\ \rangle \\
 & x \in S \;\wedge\; y = x \\
= & \langle\ (3.84)(a)\ \rangle \\
 & y \in S \;\wedge\; y = x \\
= & \langle\ (1.2)\ \rangle \\
 & y \in S \;\wedge\; x = y \\
= & \langle\ \text{Part (a)}\ \rangle \\
 & \langle y,x\rangle \in \imath_S
\end{array}
$$

For transitivity, take x, y, z to be fresh. By (9.16) it suffices to prove

$$\langle x,y\rangle \in \imath_S \;\wedge\; \langle y,z\rangle \in \imath_S \;\;\Rightarrow\;\; \langle x,z\rangle \in \imath_S\ .$$

$$
\begin{array}{cl}
 & \langle x,y\rangle \in \imath_S \;\wedge\; \langle y,z\rangle \in \imath_S \\
= & \langle\ \text{Part (a)}\ \rangle \\
 & x \in S \;\wedge\; y = x \;\wedge\; y \in S \;\wedge\; z = y \\
= & \langle\ (3.84)(a)\ \rangle \\
 & x \in S \;\wedge\; y = x \;\wedge\; y \in S \;\wedge\; z = x \\
\Rightarrow & \langle\ (3.76)(b)\ \rangle \\
 & x \in S \;\wedge\; z = x \\
= & \langle\ \text{Part (a)}\ \rangle \\
 & \langle x,z\rangle \in \imath_S
\end{array}
$$

13 Let ρ be a relation on B. Prove

$$\vdash \rho \text{ is symmetric} \quad\Rightarrow\quad (\forall n \mid n \geq 1 : \rho^n \text{ is symmetric })\ .$$

Hint: Assume the antecedent and use Mathematical Induction to prove the consequent.
Answer:
Since $\rho^1 = \rho$ is given to be symmetric, ρ^1 is symmetric.
Assume ρ^n is symmetric.
The proof that ρ^{n+1} is symmetric reduces to proving $\langle x,y\rangle \in \rho^{n+1} \;\equiv\; \langle y,x\rangle \in \rho^{n+1}$.

$$
\begin{array}{cl}
 & \langle x,y\rangle \in \rho^{n+1} \\
= & \langle\ (14.26)\ \rangle \\
 & \langle x,y\rangle \in \rho^n \circ \rho \\
= & \langle\ (14.20)\ \rangle \\
 & (\exists z \mid \langle x,z\rangle \in \rho^n \;\wedge\; \langle z,y\rangle \in \rho)
\end{array}
$$

$$
\begin{array}{ll}
= & \langle \text{ Assumption, } \rho \text{ is symmetric } \rangle \\
 & (\exists z \mid \langle z, x\rangle \in \rho^n \ \wedge \ \langle y, z\rangle \in \rho) \\
= & \langle \ (3.36) \ \rangle \\
 & (\exists z \mid \langle y, z\rangle \in \rho \ \wedge \ \langle z, x\rangle \in \rho^n) \\
= & \langle \ (14.20) \ \rangle \\
 & \langle y, x\rangle \in \rho \circ \rho^n \\
= & \langle \ (14.26) \ \rangle \\
 & \langle y, x\rangle \in \rho^{n+1} \ .
\end{array}
$$

14 Let ρ be a relation on B. For x, y, z not free in ρ,
"ρ is circular" means $(\forall x, y, z \mid \ : \langle x, y\rangle \in \rho \ \wedge \ \langle y, z\rangle \in \rho \ \Rightarrow \ \langle z, x\rangle \in \rho\ )$.
Prove,

$$\vdash \ \rho \text{ is circular } \ \wedge \ \rho \text{ is reflexive on } B \quad \Rightarrow \quad \rho \text{ is transitive} \ .$$

Answer: Assume $\rho \subseteq B \times B \ \wedge \ \rho$ is circular $\ \wedge \ \rho$ is reflexive on B . Take x, y, z to be fresh. By (9.16) it suffices to prove

$$\langle x, y\rangle \in \rho \ \wedge \ \langle y, z\rangle \in \rho \ \Rightarrow \ \langle x, z\rangle \in \rho \ .$$

$$
\begin{array}{ll}
 & \langle x, y\rangle \in \rho \ \wedge \ \langle y, z\rangle \in \rho \\
\Rightarrow & \langle \text{ Temporary Theorem: } \rho \subseteq B \times B \ \rangle \\
 & \langle z, x\rangle \in \rho \ \wedge \ \rho \subseteq B \times B \\
\Rightarrow & \langle \text{ Lemma: } \vdash \langle z, x\rangle \in \rho \ \wedge \ \rho \subseteq B \times B \ \Rightarrow x \in B, \text{ (3.60), (3.76)(b) } \rangle \\
 & \langle z, x\rangle \in \rho \ \wedge \ x \in B \\
\Rightarrow & \langle \ \rho \text{ is reflexive on } B, \text{ (4.3) } \rangle \\
 & \langle z, x\rangle \in \rho \ \wedge \ \langle x, x\rangle \in \rho \\
\Rightarrow & \langle \ \rho \text{ is circular } \rangle \\
 & \langle x, z\rangle \in \rho
\end{array}
$$

15 If the expression below is a theorem, write "This is a theorem.", and prove it. If it is not a theorem, write "This is not a theorem.", and explain in detail how you know it is not a theorem.

$(\rho \text{ is transitive}) \ \equiv \ \rho \circ \rho = \rho$

Answer: Note that if $\rho \circ \rho = \rho$, then for $n \geq 1$ it follows that $\rho^n = \rho$ from which transitivity follows. To show that the expression is **not a theorem** we need to exhibit a transitive relation ρ for which $\rho^2 \neq \rho$.

Consider the transitive relation ρ given by $\rho = \{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 0, 2\rangle\}$. We have $\rho^2 = \{\langle 0, 2\rangle\} \neq \rho$.

16 Using

Axiom 1 $\langle x, f.x\rangle \in f \quad\equiv\quad x \in \mathit{Dom}.f$,

prove that

$$\vdash\ (x \in \mathit{Dom}.f\ \wedge\ f.x \in \mathit{Dom}.g) \Rightarrow (g.(f.x) = (g \bullet f).x)\ .$$

Use the Deduction Theorem (assume the antecedent) and be careful in your treatment of expressions of the form $f.x$, i.e., don't just copy the proof that the test has on p.281 and assume that it is sufficient.

Answer:

Assume $x \in \mathit{Dom}.f$ so that $\langle x, f.x\rangle \in f$ is a temporary theorem.

Similarly, assume $f.x \in \mathit{Dom}.g$ so that $\langle f.x, g.(f.x)\rangle \in g$ is a temporary theorem.

We will prove $g.(f.x) = (g \bullet f).x$ by proving $y = g.(f.x) \quad\equiv\quad y = (g \bullet f).x$.

Observe that by (9.28) and (14.20), $\langle x, f.x\rangle \in f \wedge \langle f.x, g.(f.x)\rangle \in g \quad\Rightarrow\quad \langle x, g.(f.x)\rangle \in f \circ g$, hence by Modus Ponens, $\langle x, g.(f.x)\rangle \in f \circ g$ is a temporary theorem.

As $f \circ g \equiv g \bullet f$, this gives $\langle x, g.(f.x)\rangle \in g \bullet f$ from which $x \in \mathit{Dom}.(g \bullet f)$ is a temporary theorem.

We gave the following **Lemma** in class.

$$\vdash f \text{ is a function } \Rightarrow (\langle x, y\rangle \in f \ \equiv\ (x \in \mathit{Dom}.f\ \wedge\ y = f.x))\ .$$

A proof was given using Mutual Implication. Assume f is a function.

$$\begin{array}{ll}
 & x \in \mathit{Dom}.f\ \wedge\ y = f.x \\
= & \quad\langle \text{ Axiom 1 } \rangle \\
 & \langle x, f.x\rangle \in f\ \wedge\ y = f.x \\
= & \quad\langle\ (3.84)(a)\ \rangle \\
 & \langle x, y\rangle \in f\ \wedge\ y = f.x \\
\Rightarrow & \quad\langle\ (3.76)(b)\ \rangle \\
 & \langle x, y\rangle \in f
\end{array}$$

Now,

$$\begin{array}{ll}
 & \langle x, y\rangle \in f \\
= & \quad\langle\ (9.28), (3.60)\ \rangle \\
 & \langle x, y\rangle \in f\ \wedge\ (\exists y \mid : \langle x, y\rangle \in f) \\
= & \quad\langle\ (14.16), (3.60), \text{Axiom } 1\ \rangle \\
 & \langle x, y\rangle \in f\ \wedge\ \langle x, f.x\rangle \in f\ \wedge\ x \in \mathit{Dom}.f \\
\Rightarrow & \quad\langle\ f \text{ is definite, } (4.3)\ \rangle \\
 & y = f.x\ \wedge\ x \in \mathit{Dom}.f \\
= & \quad\langle\ (3.36)\ \rangle \\
 & x \in \mathit{Dom}.f\ \wedge\ y = f.x
\end{array}$$

The proof of the exercise now proceeds,

$$\begin{array}{ll}
 & y = (g \bullet f).x \\
= & \quad \langle \text{ Lemma: } g \bullet f \text{ is a function, } x \in \mathit{Dom}.(g \bullet f) \;\equiv\; \mathit{true} \,\rangle \\
 & \langle x, y \rangle \in g \bullet f \\
= & \quad \langle \text{ Notation } \rangle \\
 & \langle x, y \rangle \in f \circ g \\
= & \quad \langle \text{ (14.20) } \rangle \\
 & (\exists c \mid : \langle x, c \rangle \in f \;\wedge\; \langle c, y \rangle \in g) \\
= & \quad \langle \text{ Lemma: } f \text{ is a function, } x \in \mathit{Dom}.f \;\equiv\; \mathit{true} \,\rangle \\
 & (\exists c \mid : c = f.x \;\wedge\; \langle c, y \rangle \in g) \\
= & \quad \langle \text{ (9.17) } \rangle \\
 & (\exists c \mid c = f.x : \langle c, y \rangle \in g) \\
= & \quad \langle \text{ (8.14) } \rangle \\
 & \langle f.x, y \rangle \in g \\
= & \quad \langle \text{ Lemma: } g \text{ is a function, } f.x \in \mathit{Dom}.g \;\equiv\; \mathit{true} \,\rangle \\
 & y = g.(f.x)
\end{array}$$

17 The identity relation $\imath_S$ on S is defined as

$$\{x \mid x \in S : \langle x, x \rangle\} \; .$$

Carefully prove that $\imath_S$ is a total function. **Answer:** We need to prove that $\imath_S$ is determinate and that it is a total function on S.
To prove that $\imath_S$ is determinate, choose x, y to be fresh variables. By (9.16) it suffices to prove

$$\vdash \langle x, y \rangle \in \imath_S \;\wedge\; \langle x, z \rangle \in \imath_S \;\Rightarrow\; y = z \; .$$

$$\begin{array}{ll}
 & \langle x, y \rangle \in \imath_S \;\wedge\; \langle x, z \rangle \in \imath_S \\
= & \Big\langle \text{ Part (a) } \Big\rangle \\
 & x \in S \;\wedge\; y = x \;\wedge\; x \in S \;\wedge\; z = x \\
\Rightarrow & \Big\langle \text{ (3.76)(b) } \Big\rangle \\
 & y = x \;\wedge\; z = x \\
= & \Big\langle \text{ (3.84)(a) } \Big\rangle \\
 & y = z \;\wedge\; z = x \\
\Rightarrow & \Big\langle \text{ (3.76)(b) } \Big\rangle \\
 & y = z
\end{array}$$

To prove that $\imath_S$ is total on S we need to prove $\mathit{Dom}.\imath_S = S$. Choose x, y to be a fresh variables. By (9.16) it suffices to prove

$$\vdash x \in S \equiv x \in \mathit{Dom}.\imath_S \; .$$

$$\begin{array}{ll}
 & x \in \mathit{Dom}.\imath_S \\
= & \Big\langle \text{ (14.16) } \Big\rangle \\
 & x \in \{x \mid (\exists y \mid : \langle x, y \rangle \in S)\}
\end{array}$$

$$
\begin{array}{cl}
= & \langle\ (11.7)\ \rangle \\
 & (\exists y \mid\ : \langle x, y\rangle \in S) \\
= & \langle\ \text{Part (a)}\ \rangle \\
 & (\exists y \mid\ : x \in S\ \wedge\ y = x) \\
= & \langle\ (9.19)\ \rangle \\
 & (\exists y \mid y = x : x \in S) \\
= & \langle\ (8.14),\ y \text{ d.n.o.f. in } x\ \rangle \\
 & x \in S\ .
\end{array}
$$

18 You are given, with u not occurring free in S, that $\rho = \{u \mid u \in S : \langle u, v\rangle\}$.

(a) Prove,

$$\vdash \rho \text{ is a function }.$$

Answer: Take x, y, z to be fresh variables. We need to show that ρ is definite, i.e.,

$$\vdash (\forall x, y, z \mid\ : \langle x, y\rangle \in \rho\ \wedge\ \langle x, z\rangle \in \rho\quad\Rightarrow\quad y = z\ .$$

By (9.16) for multiples it suffices to prove

$$\vdash \langle x, y\rangle \in \rho\ \wedge\ \langle x, z\rangle \in \rho\quad\Rightarrow\quad y = z\ .$$

For u not ocurring free in S, consider the following

$$\textbf{Lemma}: \ \vdash\ \langle x, y\rangle \in \{u \mid u \in S : \langle u, v\rangle\}\ \ \equiv\ \ x \in S\ \wedge\ y = v\ .$$

$$
\begin{array}{cl}
 & \langle x, y\rangle \in \{u \mid u \in S : \langle u, v\rangle\} \\
= & \langle\ (11.3)\ \rangle \\
 & (\exists u \mid u \in S : \langle x, y\rangle = \langle u, v\rangle) \\
= & \langle\ (14.4)\ \rangle \\
 & (\exists u \mid u \in S : x = u\ \wedge\ y = v) \\
= & \langle\ (9.19),\ (1.3)\ \rangle \\
 & (\exists u \mid x = u :\ u \in S \wedge\ y = v) \\
= & \langle\ (8.14)\ \rangle \\
 & x \in S \wedge\ y = v
\end{array}
$$

Now,

$$
\begin{array}{cl}
 & \langle x, y\rangle \in \rho\ \wedge\ \langle x, z\rangle \in \rho \\
= & \langle\ \text{Lemma}\ \rangle \\
 & x \in S\ \wedge\ y = v\ \wedge\ x \in S\ \wedge\ z = v \\
\Rightarrow & \langle\ (3.76)\text{(b)}\ \rangle \\
 & y = v\ \wedge\ z = v
\end{array}
$$

$$\begin{array}{cl} = & \langle\ (3.84)(a)\ \rangle \\ & y = z\ \wedge\ z = v \\ \Rightarrow & \langle\ (3.76)(b)\ \rangle \\ & y = z \end{array}$$

(b) Prove,

$$\vdash \text{Dom} \,.\, \rho = S\ .$$

Answer: Take y, z to be fresh variables. By (11.4) and (9.16) it suffices to prove

$$\vdash z \in \text{Dom}.\rho\ \equiv\ z \in S\ .$$

$$\begin{array}{cl} & z \in \text{Dom}.\rho \\ = & \langle\ (14.16)\ \rangle \\ & (\exists y \mid\ : \langle z, y\rangle \in \rho) \\ = & \langle\ \text{Lemma from Part (a)}\ \rangle \\ & (\exists y \mid\ : z \in S\ \wedge\ y = v) \\ = & \langle\ (9.19)\ \rangle \\ & (\exists y \mid y = v : z \in S) \\ = & \langle\ (8.14)\ \rangle \\ & z \in S \end{array}$$

19 Let σ and ρ be relations. Prove

$$\vdash \sigma \text{ is definite } \wedge \rho \text{ is definite } \quad \Rightarrow \quad \sigma \circ \rho \text{ is definite }\ .$$

Answer: Assume σ is definite $\wedge\, \rho$ is definite . Take x, y, z, u, v, w to be fresh variables.

$$\begin{array}{cl} & \sigma \circ \rho \text{ is definite} \\ = & \langle\ (14.37),\ (9.2)\ \rangle \\ & (\forall x, y, z \mid\ : \langle x, y\rangle \in \sigma \circ \rho\ \wedge\ \langle x, z\rangle \in \sigma \circ \rho\ \ \Rightarrow\ \ y = z)\ . \end{array}$$

As x, y, z do not occur free in the Assumption, we can apply Strong(9.16). It suffices to prove

$$\langle x, y\rangle \in \sigma \circ \rho\ \wedge\ \langle x, z\rangle \in \sigma \circ \rho\ \ \Rightarrow\ \ y = z\ .$$

$$\begin{array}{cl} & \langle x, y\rangle \in \sigma \circ \rho\ \wedge\ \langle x, z\rangle \in \sigma \circ \rho\ \ \Rightarrow\ \ y = z \\ = & \langle\ (14.20),\ (3.65)\ \rangle \\ & (\exists w \mid\ : \langle x, w\rangle \in \sigma\ \wedge\ \langle w, y\rangle \in \rho)\ \ \Rightarrow\ \ (\langle x, z\rangle \in \sigma \circ \rho\ \Rightarrow\ y = z)\ . \end{array}$$

As w does not occur free in the Assumption, by Strong (9.30) it suffices to prove

$$\langle x, u\rangle \in \sigma\ \wedge\ \langle u, y\rangle \in \rho\ \ \Rightarrow\ \ (\langle x, z\rangle \in \sigma \circ \rho\ \Rightarrow\ y = z)\ .$$

$$\begin{aligned}
&\langle x,u\rangle \in \sigma \ \wedge\ \langle u,y\rangle \in \rho \ \Rightarrow\ (\langle x,z\rangle \in \sigma \circ \rho \ \Rightarrow\ y=z) \\
=\ &\big\langle \text{ (14.20), (3.65) } \big\rangle \\
&(\exists w \mid\ : \langle x,w\rangle \in \sigma \ \wedge\ \langle w,z\rangle \in \rho) \ \Rightarrow\ (\langle x,u\rangle \in \sigma \ \wedge\ \langle u,y\rangle \in \rho \ \Rightarrow\ y=z)\ .
\end{aligned}$$

As w does not occur free in the Assumption, by Strong (9.30) it suffices to prove

$$\langle x,v\rangle \in \sigma \ \wedge\ \langle v,z\rangle \in \rho \ \Rightarrow\ (\langle x,u\rangle \in \sigma \ \wedge\ \langle u,y\rangle \in \rho \ \Rightarrow\ y=z)\ .$$

$$\begin{aligned}
&\langle x,v\rangle \in \sigma \ \wedge\ \langle v,z\rangle \in \rho \ \Rightarrow\ (\langle x,u\rangle \in \sigma \ \wedge\ \langle u,y\rangle \in \rho \ \Rightarrow\ y=z) \\
=\ &\big\langle \text{ 3.65) } \big\rangle \\
&\langle x,v\rangle \in \sigma \ \wedge\ \langle v,z\rangle \in \rho \ \wedge\ \langle x,u\rangle \in \sigma \ \wedge\ \langle u,y\rangle \in \rho \ \Rightarrow\ y=z \\
=\ &\big\langle \text{ (3.36), (3.37) } \big\rangle \\
&\langle x,v\rangle \in \sigma \ \wedge\ \langle x,u\rangle \in \sigma \ \wedge\ \langle u,y\rangle \in \rho \ \wedge\ \langle v,z\rangle \in \rho \ \Rightarrow\ y=z \\
=\ &\big\langle \text{ Temporary Theorem: } \sigma \text{ is definite } \big\rangle \\
&\langle x,v\rangle \in \sigma \ \wedge\ \langle x,u\rangle \in \sigma \ \wedge\ v=u \ \wedge\ \langle u,y\rangle \in \rho \ \wedge\ \langle v,z\rangle \in \rho \ \Rightarrow\ y=z \\
=\ &\big\langle \text{ (3.84)(a) } \big\rangle \\
&\langle x,v\rangle \in \sigma \ \wedge\ \langle x,u\rangle \in \sigma \ \wedge\ v=u \ \wedge\ \langle u,y\rangle \in \rho \ \wedge\ \langle u,z\rangle \in \rho \ \Rightarrow\ y=z \\
=\ &\big\langle \text{ Temporary Theorem: } \rho \text{ is definite } \big\rangle \\
&\langle x,v\rangle \in \sigma \ \wedge\ \langle x,u\rangle \in \sigma \ \wedge\ v=u \ \wedge\ \langle u,y\rangle \in \rho \ \wedge\ \langle u,z\rangle \in \rho \ \wedge\ y=z \ \Rightarrow\ y=z
\end{aligned}$$

which is (3.86)(b).

20 Let B be a nonempty set and ρ be an equivalence relation on B.

(a) Prove that there exists a total function f satisfying

$Dom.f = B$.
f is onto.
$\langle b,c\rangle \in \rho$ if and only if $f.b = f.c$.

Note: To answer the question, make an appropriate choice for $Ran.f$.
Answer: Let $C = \{b : B \mid :\ [b]_\rho\}$, the set of equivalence classes of ρ. Define $f : B \to C$ by $f.b = [b]_\rho$.

$$\begin{aligned}
&f.b = f.c \\
=\ &\quad\langle \text{ defintion of } f \ \rangle \\
&[b]_\rho = [c]_\rho \\
=\ &\quad\langle \text{ (14.35) } \rangle \\
&b\,\rho\,c \\
=\ &\quad\langle \text{ definition of } b\,\rho\,c \ \rangle \\
&\langle b,c\rangle \in \rho
\end{aligned}$$

(b) With f as in (a), prove if g is any total function with $Dom.g = B$ satisfying $g.b = g.c$ whenever $\langle b,c\rangle \in \rho$, there exists a unique total function h with $Dom.h = Ran.f$ such that $h \bullet f = g$.
Answer: Let $h = \{b : B \mid\ : \langle [b]_\rho, g.b\rangle\}$. Once h is proved to be determinate, we know h is a function, and since $h.[b]_\rho = g.b$, we have $h \bullet f = g$.
But by (14.35), $[b]_\rho = [c]_\rho \ \equiv\ b\,\rho\,c$, from which , $[b]_\rho = [c]_\rho \ \Rightarrow\ g.b = g.c$.