Chapter 14 Exercise List

1. (a) i. Prove,

$$\vdash (\exists y \mid y \in T : z = \langle s, y\rangle) \;\Rightarrow\; (+y \mid z = \langle s, y\rangle : 1) = 1 \ .$$

ii. Fill in reasons and complete the following proof that

$$\vdash \#(\{s\} \times T) \;=\; \#T \ .$$

Let $z,\ x,\ y,\ w$ be fresh variables. We are using $=$ for **type** $\mathbb{N}$ in the left margin.

$$
\begin{aligned}
& \#(\{s\} \times T) \\
= \;& \langle \qquad\qquad\qquad \rangle \\
& (+z \mid z \in \{s\} \times T : 1) \\
= \;& \langle \qquad\qquad\qquad \rangle \\
& (+z \mid (\exists x, y \mid x \in \{s\} \wedge y \in T : z = \langle x, y\rangle) : 1) \\
= \;& \langle \qquad\qquad\qquad \rangle \\
& (+z \mid (\exists x, y \mid x = s \wedge y \in T : z = \langle x, y\rangle) : 1) \\
= \;& \langle \qquad\qquad\qquad \rangle \\
& (+z \mid (\exists x \mid x = s : (\exists y \mid y \in T : z = \langle x, y\rangle)) : 1) \\
= \;& \langle \qquad\qquad\qquad \rangle \\
& (+z \mid (\exists y \mid y \in T : z = \langle s, y\rangle) : 1) \\
= \;& \langle \qquad\qquad\qquad \rangle \\
& (+z \mid (\exists y \mid y \in T : z = \langle s, y\rangle) : (+y \mid z = \langle s, y\rangle : 1)) \\
= \;& \langle \qquad\qquad\qquad \rangle \\
& (+z \mid (\exists w \mid w \in T : z = \langle s, w\rangle) : (+y \mid z = \langle s, y\rangle : 1)) \\
= \;& \langle \qquad\qquad\qquad \rangle \\
& \dots
\end{aligned}
$$

(b) Prove,

$$\vdash (s \neq s') \;\Rightarrow\; (\,\#(\{s, s'\} \times T) \;=\; \#T + \#T\,) \ .$$

2. Prove,

$$\vdash \text{Dom}.(\rho \circ \sigma) \subseteq \text{Dom}.\rho \ .$$

3. Prove, provided x, y do not have free occurrences in S or T,

$$\vdash S \neq \emptyset \wedge T \neq \emptyset \Rightarrow (\exists x, y \;\mid : \langle x, y\rangle \in S \times T) \ .$$

4. Prove,

$$\vdash (S \times T \;\cap\; T \times S) = \emptyset \;\Rightarrow\; T \cap S = \emptyset \ .$$

5. Prove,

$$\vdash S \neq \emptyset \;\wedge\; T \neq \emptyset \;\equiv\; S \times T \neq \emptyset \ .$$

6. Prove that

$$\vdash \;S \times T \;\subseteq\; S \times U \quad\Rightarrow\quad T \subseteq U \;\vee\; S = \emptyset \ .$$

7. (a) Prove,

$$\vdash \langle x, y \rangle \in \iota_B \quad \equiv \quad x \in B \ \wedge \ x = y \ .$$

(b) Prove,

$$\vdash \iota_B^{-1} = \iota_B \ .$$

(c) Let ρ be a relation. Prove,

$$\vdash \rho \subseteq B \times B \quad \Rightarrow \quad \iota_B \circ \rho = \rho \ .$$

8. (a) Prove that provided z d.n.o.f. Q, $\vdash (\forall z \,|\, P : Q) \ \equiv \ ((\exists z \,|\, : P) \Rightarrow Q$.

(b) Prove, $\vdash \ \rho \subseteq B \times C \quad \Rightarrow \quad Dom.\rho \subseteq B$.

9. Let ρ be a relation, and suppose that n does not occur freely in ρ .
Use Mathematical Induction to prove that $\vdash \ (\forall n : \mathbb{N} \,|\, 1 \leq n : \text{ Dom. } \rho^n \subseteq \text{Dom. } \rho)$.

10. Let ρ be a relation on S which is reflexive on S and transitive. Prove that $\rho = \rho^2$.

11. Assume that ρ is a binary relation on a non-empty set S. Decide which of the following are theorems. Prove or justify your answers.

(a) $r(\rho)$ is symmetric.

(b) ρ^2 is transitive.

(c) $s(r(\rho)) = r(s(\rho))$.

(d) $(s(\rho))^+$ is reflexive on S.

12. Recall that for x not occurring free in S, $\iota_S = \{x \,|\, x \in S : \langle x, x \rangle\}$.

(a) Prove,

$$\vdash \langle x, y \rangle \in \iota_S \quad \equiv \quad x \in S \ \wedge \ y = x \ .$$

(b) Prove,

$$\vdash \iota_S \text{ is an equivalence relation on } S \ .$$

13. Let ρ be a relation on B. Prove

$$\vdash \rho \text{ is symmetric} \quad \Rightarrow \quad (\forall n \,|\, n \geq 1 : \rho^n \text{ is symmetric }) \ .$$

Hint: Assume the antecedent and use Mathematical Induction to prove the consequent.

14. Let ρ be a relation on B. For x, y, z not free in ρ,
"ρ is circular" means $(\forall x, y, z \,|\, : \langle x, y \rangle \in \rho \ \wedge \ \langle y, z \rangle \in \rho \quad \Rightarrow \quad \langle z, x \rangle \in \rho)$.
Prove,

$$\vdash \rho \text{ is circular } \ \wedge \ \rho \text{ is reflexive on } B \quad \Rightarrow \quad \rho \text{ is transitive} \ .$$

15. If the expression below is a theorem, write "This is a theorem.", and prove it. If it is not a theorem, write "This is not a theorem.", and explain in detail how you know it is not a theorem.

$(\rho \text{ is transitive}) \quad \equiv \quad \rho \circ \rho = \rho$

16. Using

    Axiom 1 $\langle x, f.x \rangle \in f \quad \equiv \quad x \in Dom.f$,

    prove that
    $$\vdash (x \in Dom.f \ \wedge \ f.x \in Dom.g) \Rightarrow (g.(f.x) = (g \bullet f).x) \ .$$
    Use the Deduction Theorem (assume the antecedent) and be careful in your treatment of expressions of the form $f.x$, i.e., don't just copy the proof that the test has on p.281 and assume that it is sufficient.

17. The identity relation $\imath_S$ on S is defined as
    $$\{x \,|\, x \in S : \langle x, x \rangle\} \ .$$
    Carefully prove that $\imath_S$ is a total function.

18. You are given, with u not occurring free in S, that $\rho = \{u \,|\, u \in S : \langle u, v \rangle\}$.

    (a) Prove,
    $$\vdash \rho \text{ is a function } .$$
    (b) Prove,
    $$\vdash \text{Dom} \,.\, \rho = S \ .$$

19. Let σ and ρ be relations. Prove
    $$\vdash \sigma \text{ is definite } \wedge \rho \text{ is definite } \quad \Rightarrow \quad \sigma \circ \rho \text{ is definite } \ .$$

20. Let B be a nonempty set and ρ be an equivalence relation on B.

    (a) Prove that there exists a total function f satisfying

        $Dom.f = B$.

        f is onto.

        $\langle b, c \rangle \in \rho$ if and only if $f.b = f.c$.

        Note: To answer the question, make an appropriate choice for $Ran.f$.

    (b) With f as in (a), prove if g is any total function with $Dom.g = B$ satisfying $g.b = g.c$ whenever $\langle b, c \rangle \in \rho$, there exists a unique total function h with $Dom.h = Ran.f$ such that $h \bullet f = g$.