York University

Atkinson College, Faculty of Arts, Faculty of Science

Math 2090

Midterm Examination

SOLUTIONS

Instructions:

1. Time allowed : 90 minutes.
2. There are 6 questions on 8 pages.
3. Answer all questions.
4. Prove, means prove using the proof methods and proof format of the text.
   If your reason is not a theorem on the list provided, or another question on this test, you must provide its proof.
5. Your reasons must indicate that you have verified any conditions concerning free occurrences of variables.

Question	Points	Marks
1	3	
2	5	
3	4	
4	7	
5	4	
6	7	
Total	30	

1. (3 points) Is the following a correct proof of $S \subseteq \{x\}$?
If it is correct, provide complete reasons for each step.
If it is not correct, find and explain all errors in the proof.

Take x to be a fresh variable.

$$
\begin{array}{ll}
 & S \subseteq \{x\} \\
= & \langle \star\star \qquad\qquad \rangle \\
 & (\forall x \mid x \in S : x \in \{x\}) \\
= & \langle \qquad\qquad \rangle \\
 & (\forall x \mid x \in S : true) \\
= & \langle \qquad\qquad \rangle \\
 & (\forall x \mid \ : x \in S \Rightarrow true) \\
= & \langle \qquad\qquad \rangle \\
 & (\forall x \mid \ : true) \\
= & \langle \qquad\qquad \rangle \\
 & true\ .
\end{array}
$$

Answer: The proof is not correct. It has two errors.

First, as x occurs in the expression $S \subseteq \{x\}$, it cannot be declared as a fresh variable.

In addition, the step with reasons marked $\star\star$ is not correct. There is an apparent attempt to use (11.13). This is incorrect as x occurs free in $\{x\}$.

In any case, consider any particular interpretation of the set S and variable x where $S \subseteq \{x\}$ interprets False. For example, one might take S to be the set consisting of the number 5 and x to be the number 3. The first step, $S \subseteq \{x\} \quad \equiv \quad (\forall x \mid x \in S : x \in \{x\})$, interprets False, and therefore cannot be a theorem.

2. (5 points) Prove,

$$\vdash \{x,y\} = \{z\} \quad \equiv \quad x = z \wedge y = z \ .$$

Answer: Take w to be a fresh variable.

$$
\begin{array}{ll}
 & \{x,y\} = \{z\} \\
= & \langle\ (11.4)\ \rangle \\
 & (\forall w \mid : w \in \{x,y\} \quad \equiv \quad w \in \{z\}) \\
= & \langle\ \text{Abbreviation}\ \rangle \\
 & (\forall w \mid : w \in \{w \mid w = x \vee w = y\} \quad \equiv \quad w \in \{w \mid w = z\}) \\
= & \langle\ (11.7)\ \rangle \\
 & (\forall w \mid : w = x \vee w = y \quad \equiv \quad w = z) \\
= & \langle\ (9.13),\ (3.60)\ \rangle \\
 & (\forall w \mid : w = x \vee w = y \quad \equiv \quad w = z) \wedge (x = x \vee x = y \quad \equiv \quad x = z) \\
 & \wedge\, (y = x \vee y = y \quad \equiv \quad y = z) \\
= & \langle\ (1.2),\ (3.29),\ (3.3)\ \rangle \\
 & (\forall w \mid : w = x \vee w = y \quad \equiv \quad w = z) \wedge x = z \wedge y = z \\
= & \langle\ (3.84)(\text{a})\ \rangle \\
 & (\forall w \mid : w = z \vee w = y \quad \equiv \quad w = z) \wedge x = z \wedge y = z \\
= & \langle\ (3.84)(\text{a})\ \rangle \\
 & (\forall w \mid : w = z \vee w = z \quad \equiv \quad w = z) \wedge x = z \wedge y = z \\
= & \langle\ (3.26)\ \rangle \\
 & (\forall w \mid : \mathit{true}) \wedge x = z \wedge y = z \\
= & \langle\ (9.8)\ \rangle \\
 & \mathit{true} \wedge x = z \wedge y = z \\
= & \langle\ (3.39)\ \rangle \\
 & x = z \wedge y = z
\end{array}
$$

3. (4 points) Prove,

$$\vdash S \subseteq T \equiv T = S \cup (T - S) .$$

Answer: Take w to be a fresh variable.

$$\begin{array}{ll}
 & T = S \cup (T - S) \\
= & \langle\ (1.4)\ \rangle \\
 & (\forall w \mid : w \in T \equiv w \in S \cup (T - S)) \\
= & \langle\ (11.20), (11.22)\ \rangle \\
 & (\forall w \mid : w \in T \equiv w \in S \vee (w \in T \wedge w \notin S)) \\
= & \langle\ (3.44)(b)\ \rangle \\
 & (\forall w \mid : w \in T \equiv w \in S \vee w \in T) \\
= & \langle\ (3.57)\ \rangle \\
 & (\forall w \mid : w \in S \Rightarrow w \in T) \\
= & \langle\ (9.2)\ \rangle \\
 & (\forall w \mid w \in S : w \in T) \\
= & \langle\ (11.13)\ \rangle \\
 & S \subseteq T
\end{array}$$

4. (a) (3 points) Prove (11.57),

$$\vdash S \subseteq T \wedge T \subseteq S \quad \equiv \quad S = T \ .$$

Answer: Take w to be a fresh variable.

$$
\begin{array}{cl}
 & S \subseteq T \wedge T \subseteq S \\
= & \langle \ (11.13) \ \rangle \\
 & (\forall w \mid w \in S : w \in T) \ \wedge \ (\forall w \mid w \in T : w \in S) \\
= & \langle \ (9.2) \ \rangle \\
 & (\forall w \mid \ : w \in S \ \Rightarrow \ w \in T) \ \wedge \ (\forall w \mid \ : w \in T \ \Rightarrow \ w \in S) \\
= & \langle \ (8.15) \ \rangle \\
 & (\forall w \mid \ : (w \in S \ \Rightarrow \ w \in T) \ \wedge \ (w \in T \ \Rightarrow \ w \in S)) \\
= & \langle \ (3.80) \ \rangle \\
 & (\forall w \mid \ : w \in S \ \equiv \ w \in T) \\
= & \langle \ (11.4) \ \rangle \\
 & S = T
\end{array}
$$

(b) (4 points) Prove,

$$\vdash S \subset \{x \mid R\} \quad \Rightarrow \quad (\exists x \mid \, : R) \; .$$

Answer: This proof exploits Part (a).
A proof using Contrapositive and Deduction can be found in the homework solutions.
Take w to be a fresh variable.

$$
\begin{array}{ll}
 & S \subset \{x \mid R\} \\
= & \langle \; (11.14) \; \rangle \\
 & S \subseteq \{x \mid R\} \;\wedge\; \neg(S = \{x \mid R\}) \\
= & \langle \; \text{Part (a)} \; \rangle \\
 & S \subseteq \{x \mid R\} \;\wedge\; \neg(S \subseteq \{x \mid R\} \;\wedge\; \{x \mid R\} \subseteq S) \\
= & \langle \; (3.47)(a) \; \rangle \\
 & S \subseteq \{x \mid R\} \;\wedge\; (S \not\subseteq \{x \mid R\} \;\vee\; \{x \mid R\} \not\subseteq S) \\
= & \langle \; (3.44)(a) \; \rangle \\
 & S \subseteq \{x \mid R\} \;\wedge\; \{x \mid R\} \not\subseteq S \\
\Rightarrow & \langle \; (3.76)(b) \; \rangle \\
 & \neg(\{x \mid R\} \subseteq S) \\
= & \langle \; (11.13) \; \rangle \\
 & \neg(\forall w \mid w \in \{x \mid R\} : w \in S) \\
= & \langle \; (9.18)(c), (9.19) \; \rangle \\
 & (\exists w \mid \, : w \in \{x \mid R\} \;\wedge\; w \notin S) \\
\Rightarrow & \langle \; (3.76)(b), (9.16), (9.17), \text{Modus Ponens} \; \rangle \\
 & (\exists w \mid \, : w \in \{x \mid R\}) \\
= & \langle \; (8.21) \; \rangle \\
 & (\exists x \mid \, : x \in \{x \mid R\}) \\
= & \langle \; (11.7) \; \rangle \\
 & (\exists x \mid \, : R)
\end{array}
$$

5. (4 points) Prove,

$$\vdash T = \emptyset \;\Rightarrow\; (S \cup T) - (S \cap T) = S \; .$$

Answer:

$$\begin{array}{ll}
& T = \emptyset \;\Rightarrow\; (S \cup T) - (S \cap T) = S \\
= & \langle \text{ (3.84)(b) } \rangle \\
& T = \emptyset \;\Rightarrow\; (S \cup \emptyset) - (S \cap \emptyset) = S \\
= & \left\langle \begin{array}{l} \text{Lemma: } \vdash S \cup \emptyset = S \\ \text{Lemma: } \vdash S \cap \emptyset = \emptyset \end{array} \right\rangle \\
& T = \emptyset \;\Rightarrow\; S - \emptyset = S \\
= & \langle \text{ Lemma: } \vdash S - \emptyset = S \rangle \\
& T = \emptyset \;\Rightarrow\; S = S \\
= & \langle \text{ (1.2) } \rangle \\
& T = \emptyset \;\Rightarrow\; true \\
= & \langle \text{ (3.72) } \rangle \\
& true
\end{array}$$

For $\vdash \; S \cup \emptyset \; = \; S$, take w fresh and observe that by (11.4) and (9.16) it suffices to prove $w \in S \cup \emptyset \;\; \equiv \;\; w \in S$.

$$\begin{array}{ll}
& w \in S \cup \emptyset \\
= & \langle \text{ (11.20) } \rangle \\
& w \in S \vee w \in \emptyset \\
= & \langle \text{ Definition of } \emptyset \rangle \\
& w \in S \vee false \\
= & \langle \text{ (3.30) } \rangle \\
& w \in S
\end{array}$$

For $\vdash \; S \cap \emptyset \; = \; \emptyset$, take w fresh and observe that by (11.4) and (9.16) it suffices to prove $w \in S \cap \emptyset \;\; \equiv \;\; w \in \emptyset$.

$$\begin{array}{ll}
& w \in S \cap \emptyset \\
= & \langle \text{ (11.21) } \rangle \\
& w \in S \wedge w \in \emptyset \\
= & \langle \text{ Definition of } \emptyset \rangle \\
& w \in S \wedge false \\
= & \langle \text{ (3.40) } \rangle
\end{array}$$

$$\begin{array}{ll} & false \\ = & \langle \text{ Definition of } \emptyset \rangle \\ & w \in \emptyset \end{array}$$

For $\vdash\ S - \emptyset \ = \ S$, take w fresh and observe that by (11.4) and (9.16) it suffices to prove $w \in S - \emptyset \ \equiv \ w \in S$.

$$\begin{array}{ll} & w \in S - \emptyset \\ = & \langle \text{ (11.22) } \rangle \\ & w \in S \wedge \neg (w \in \emptyset) \\ = & \langle \text{ Definition of } \emptyset \rangle \\ & w \in S \wedge \neg false \\ = & \langle \text{ (3.13) } \rangle \\ & w \in S \wedge true \\ = & \langle \text{ (3.39) } \rangle \\ & w \in S \end{array}$$

6. (a) (2 points) Prove,

$$\vdash x \in S \wedge S \subseteq T \ \Rightarrow \ x \in T \ .$$

Answer: Take w to be a fresh variable.

$$\begin{array}{ll} & x \in S \wedge S \subseteq T \\ = & \langle \text{ (11.4) } \rangle \\ & x \in S \wedge (\forall w \mid w \in S : w \in T) \\ = & \langle \text{ (9.2) } \rangle \\ & x \in S \wedge (\forall w \mid \ : w \in S \ \Rightarrow \ w \in T) \\ = & \langle \text{ (9.13), (3.60) } \rangle \\ & x \in S \wedge (\forall w \mid \ : w \in S \ \Rightarrow \ w \in T) \wedge (x \in S \ \Rightarrow \ x \in T) \\ = & \langle \text{ (3.66) } \rangle \\ & x \in S \wedge x \in T \wedge (\forall w \mid \ : w \in S \ \Rightarrow \ w \in T) \\ \Rightarrow & \langle \text{ (3.76)(b) } \rangle \\ & x \in T \end{array}$$

(b) (5 points) Prove,

$$\vdash (\exists x \mid : S \subseteq \{x\}) \;\Rightarrow\; \#S = 0 \vee \#S = 1 \;.$$

Note: You may use the "Useful Lemma", $\vdash x \in \{y\} \;\equiv\; x = y$, without giving its proof.
Answer: Take w to be a fresh variable. By (9.30) it suffices to prove

$$\vdash S \subseteq \{w\} \;\Rightarrow\; \#S = 0 \vee \#S = 1 \;.$$

We prove

$$\vdash S \subseteq \{w\} \wedge w \in S \;\Rightarrow\; \#S = 1 \quad\text{and}\quad \vdash S \subseteq \{w\} \wedge w \notin S \;\Rightarrow\; \#S = 0$$

from which by (3.76)(a) and Case Analysis the result follows.

Assume $S \subseteq \{w\} \wedge w \in S$.

$$\begin{array}{ll} & S \subseteq \{w\} \wedge w \in S \\ = & \langle \text{ Lemma: } \vdash w \in S \;\equiv\; \{w\} \subseteq S \ \rangle \\ & S \subseteq \{w\} \wedge \{w\} \subseteq S \\ = & \langle \text{ Problem 4(a) } \rangle \\ & S = \{w\} \end{array}$$

gives that $S = \{w\}$ is a temporary theorem.
To prove $\vdash w \in S \;\equiv\; \{w\} \subseteq S$, assume $x \in S$. Take v to be a fresh variable.

$$\begin{array}{ll} & \{w\} \subseteq S \\ = & \langle \text{ (11.13) } \rangle \\ & (\forall v \mid v \in \{w\} : v \in S) \\ = & \langle \text{ Useful Lemma } \rangle \\ & (\forall v \mid v = w : v \in S) \\ = & \langle \text{ *.14) } \rangle \\ & w \in S \end{array}$$

Now take z to be a fresh variable.

$$\begin{array}{ll} & \#S = 1 \\ = & \langle \text{ Leibniz, } S = \{w\} \text{ is a temporary theorem } \rangle \\ & \#\{w\} = 1 \\ = & \langle \text{ (11.12) } \rangle \\ & (+z \mid z \in \{w\} : 1) = 1 \\ = & \langle \text{ Useful Lemma } \rangle \\ & (+z \mid z = w : 1) = 1 \end{array}$$

which is (8.14).

Assume $S \subseteq \{w\} \wedge w \notin S$. Take y to be a fresh variable.

$$\begin{array}{cl} & y \in S \\ = & \langle \text{ Assumption, (3.39) } \rangle \\ & y \in S \wedge S \subseteq \{w\} \wedge w \notin S \\ = & \langle \text{ Part (a) } \rangle \\ & y \in S \wedge S \subseteq \{w\} \wedge y \in \{w\} \wedge w \notin S \\ = & \langle \text{ Useful Lemma } \rangle \\ & y \in S \wedge S \subseteq \{w\} \wedge y = w \wedge w \notin S \\ = & \langle \text{ (3.84)(a) } \rangle \\ & w \in S \wedge S \subseteq \{w\} \wedge y = w \wedge w \notin S \\ = & \langle \text{ (3.42), (3.40) } \rangle \\ & false \end{array}$$

which using the Strong Version of (9.16) and (11.4) gives, $S = \emptyset$ is a temporary theorem. Take z to be a fresh variable.

$$\begin{array}{cl} & \#S = 0 \\ = & \langle \text{ Leibniz, } S = \emptyset \text{ is a temporary theorem } \rangle \\ & \#\emptyset = 0 \\ = & \langle \text{ (11.12) } \rangle \\ & (+z \mid z \in \emptyset : 1) = 0 \\ = & \langle \text{ Defintion of } \emptyset \rangle \\ & (+z \mid false : 1) = 0 \end{array}$$

which is (8.13).

The end