General Reminder: It is YOUR responsibility to know when you must be careful with “dnof” conditions, and to cite the restriction appropriately.

Some “Bonus theorems”, etc.

Theorem (repaired (8.16)): For any symmetric, associative binary connective $*$ with identity,
$\vdash \ (\forall x| : R \wedge S \ \equiv \ false) \ \Rightarrow \ ((*x| \ R \vee S : \ P) \ = \ (*x|R : P) * (*x|S : P))$.

(9.7′): If x dnof in P , then $\vdash \ (\forall x| : \ P \wedge Q) \ \equiv \ P \wedge (\forall x| : \ Q)$.

(9.23′): If x dnof in P , then $\vdash \ (\exists x| : \ P \vee Q) \ \equiv \ P \vee (\exists x| : \ Q)$.

(9.26′): $\vdash \ (\exists x| R : \ P \wedge Q) \ \Rightarrow \ (\exists x| R : \ P)$.

Chapter 11: Axioms and Theorems

(11.3) **Axiom, Set membership:** If x dnof in P , then
$\vdash \ P \in \{x \,|\, R : \ E\} \ \equiv \ (\exists x \,|\, R : \ P = E)$.

(11.4) **Axiom, Extensionality:** If x dnof in S, T , then
$\vdash \ S = T \ \equiv \ (\forall x \,| : x \in S \ \equiv \ x \in T)$.

(11.5) **Theorem:** If x dnof in S , then $\vdash \ S \ = \ \{x \,|\, x \in S\}$.

(11.6) **Theorem:** If y dnof in R, E , then $\vdash \ \{x \,|\, R : \ E\} \ = \ \{y \,|\, (\exists x \,|\, R : \ y = E)\}$.

(11.7) **Theorem:** $x \in \{x \,|\, R\} \ \equiv \ R$

(11.9) **Theorem:** $\{x \,|\, Q\} \ = \ \{x \,|\, R\} \ \equiv \ (\forall x \,| : Q \equiv R)$

(11.10) **Metatheorem:** $\vdash \ \{x \,|\, Q\} \ = \ \{x \,|\, R\}$ if and only if $\vdash \ Q \equiv R$.

(11.12) **Axiom, Size:** If x dnof in S , then $\vdash \ \#S \ = \ (\sum x \,|\, x \in S : \ 1)$.

(11.13) **Axiom, Subset:** If x dnof in S, T , then $\vdash \ S \subseteq T \ \equiv \ (\forall x \,|\, x \in S : x \in T)$.

(11.14) **Axiom, Proper subset:** $S \subset T \ \equiv \ S \subseteq T \ \wedge \ S \neq T$

(11.15) **Axiom, Superset:** $T \supseteq S \ \equiv \ S \subseteq T$

(11.16) **Axiom, Proper superset:** $T \supset S \ \equiv \ S \subset T$

(11.17) **Axiom, Complement:** $x \in \sim S \ \equiv \ x \notin S$

(11.19) **Theorem:** $\sim\sim S \ = \ S$

(11.20) **Axiom, Union:** $x \in S \cup T \ \equiv \ x \in S \ \vee \ x \in T$

(11.21) **Axiom, Intersection:** $x \in S \cap T \ \equiv \ x \in S \ \wedge \ x \in T$

(11.22) **Axiom, Difference:** $x \in S - T \ \equiv \ x \in S \ \wedge \ x \notin T$

(11.23) **Axiom, Power set:** $y \in \mathcal{P}S \ \equiv \ y \subseteq S$

You may also use the following facts without giving references for them:
(1) $\vdash \ x \in \emptyset \ \equiv$ false , and $\vdash \ x \in \mathbb{U} \ \equiv$ true .
(2) $\{e_1, \ldots, e_n\}$ *means* $\{x| \ (x = e_1) \ \vee \ \ldots \ \vee \ (x = e_n)\}$, if x dnof in $e_1, \ldots, e_n$.
(3) $\{x| \, R\}$ is an abbreviation for $\{x| \, R : \ x\}$. You may also use Dummy Renaming for set comprehensions, from class.

Chapter 12: Axioms and Theorems

(12.3) **Axiom:** $(\forall n{:}\mathbb{N} \mid : \ (\forall i \mid 0 \le i < n : P.i) \Rightarrow P.n) \quad \Rightarrow \quad (\forall n{:}\mathbb{N} \mid : \ P.n)$
(12.5) $\vdash \ P.0 \ \wedge \ (\forall n{:}\mathbb{N} \mid : \ (\forall i \mid 0 \le i \le n : P.i) \ \Rightarrow P.(n+1) \ ) \quad \Rightarrow \quad (\forall n{:}\mathbb{N} \mid : \ P.n)$.

(12.8) $\vdash \ P.n_0 \ \wedge \ (\forall n{:}\mathbb{N} \mid n_0 \le n \ : \ (\forall i \mid n_0 \le i \le n : P.i) \ \Rightarrow P.(n+1) \ ) \quad \Rightarrow \quad (\forall n{:}\mathbb{N} \mid n_0 \le n \ : \ P.n)$.

Chapter 14: Axioms, Theorems, Definitions

In the Ch. 14 list, it is understood that lower case Greek letters represent relations, lower case Latin letters are variables, and upper case letters are arbitrary given expressions. $C \rho D$ is an abbreviation for $\langle C, D\rangle \in \rho$.

(14.2′) **Axiom (pair equality):** $\langle A,B\rangle = \langle C,D\rangle \quad \equiv \quad A = C \ \wedge \ B = D$
(14.3′) **Axiom (cross product):** If x, y d.n.o.f. in E, S, T, then

$$\vdash \ E \in S \times T \ \equiv \ (\exists x, y \,|\, x \in S \ \wedge \ y \in T : E = \langle x, y\rangle) \ .$$

(14.4′) $\vdash \ \langle A,B\rangle \in S \times T \ \equiv \ A \in S \ \wedge \ B \in T$.
(14.5′) $\vdash \ \langle A,B\rangle \in S \times T \ \equiv \ \langle B,A\rangle \in T \times S$.
(14R) **Definition (relation):** Let W be a set expression. "W is a **relation (expression)**" **means:** For some variables x, y not occurring freely in W ,

$$z \in W \quad \Rightarrow \quad (\exists x, y \,| \ : z = \langle x, y\rangle) \ .$$

(14RP) **Theorem (relational property):** For any relation ρ , variables x, y not occurring freely in ρ, P , and variable z not occurring freely in ρ ,

$$\vdash \ (\forall z \,|\, z \in \rho : P) \ \equiv \ (\forall x, y \,|\, \langle x,y\rangle \in \rho : P[z := \langle x,y\rangle]) \ .$$

(14RE) **Theorem (relational equality):** For any relations ρ, σ, and variables x, y not occurring freely in either of ρ, σ,

$$\vdash \ \rho = \sigma \ \equiv \ (\forall x, y \,| \ : \langle x,y\rangle \in \rho \ \equiv \ \langle x,y\rangle \in \sigma) \ .$$

(14.16) For any relation ρ , and x, y nof in ρ , $\vdash$ $\mathrm{Dom.}\,\rho = \{x \,|\, (\exists y| : \ x \rho y)\}$.
(14.17) For any relation ρ , and x, y nof in ρ , $\vdash$ $\mathrm{Ran.}\,\rho = \{y \,|\, (\exists x| : \ x \rho y)\}$.
(14.18) $\vdash \ \langle D,E\rangle \in \rho^{-1} \quad \equiv \quad \langle E,D\rangle \in \rho$.
(14.20) For any relations ρ, σ, and variable x not occurring freely in either of ρ or σ,

$$\vdash \ \langle D,E\rangle \in \rho \circ \sigma \quad \equiv \quad (\exists x| : \ \langle D,x\rangle \in \rho \wedge \langle x,E\rangle \in \sigma) \ .$$

Axiom 14Id (Identity): For any set B , and x nof in B , $\vdash$ $\iota_B = \{x \,|\, x \in B : \ \langle x,x\rangle\}$.
(14.25) If n is of type $\mathbb{N}$ and dnof in ρ , then we have the inductive definition:

$$\vdash \ \rho^1 = \rho , \quad \text{and} \quad \vdash \ 1 \le n \quad \Rightarrow \quad \rho^{n+1} = \rho^n \circ \rho \ .$$

Classes of Relations
Defn (rel. "on B"): (ρ is a relation on B) **means** $\rho \subseteq B \times B$.
Defn 14 Refl: Suppose $\vdash$ (ρ is a relation on B). Then
(ρ is reflexive on B) **means** $(\forall x| \ x \in B : \ x \rho x)$ (x nof in ρ, B).
Defn 14 Symm: (ρ is symmetric) **means** $(\forall x, y| : x \rho y \ \equiv \ y \rho x)$ (x, y nof in ρ).
Defn 14 Trans: (ρ is transitive) **means** $(\forall x, y, z | x \rho y \ \wedge \ y \rho z : \ x \rho z)$ (x, y, z nof in ρ).
Defn (14.37) Det: (ρ is determinate) **means** $(\forall x, y, z | x \rho y \ \wedge \ x \rho z : \ y = z)$ (x, y, z nof in ρ).
(ρ is a function) **just means** (ρ is determinate).

Axiom 14FV (function value): (f is a function) $\Rightarrow$ $(\langle E, f.E\rangle \in f \ \equiv \ E \in \mathrm{Dom.}\, f)$