# Math 2090 N
# Quiz 4 Version 2 March 25, 2003
# SOLUTIONS

1. (5 points) Prove that for $\sigma, \mu$ relations,

$$\vdash \sigma \subseteq \mu \quad \Rightarrow \quad \text{Ran}.\sigma \subseteq \text{Ran}.\mu \ .$$

**Hint:** Keep your goal in mind.
**Answer:** Take $x, y, z$ to be fresh variables.

$$\begin{array}{ll}
& \sigma \subseteq \mu \\
= & \langle\ (11.13),\ z \text{ d.n.o.f. in } \sigma, \mu\ \rangle \\
& (\forall z \,|\, z \in \sigma : z \in \mu) \\
= & \langle\ 14 \text{ RP}\ \rangle \\
& (\forall x, y \,|\, \langle x, y \rangle \in \sigma : \langle x, y \rangle \in \mu) \\
= & \langle\ \vdash (\star x, y \,|\, R : P) = (\star y, x \,|\, R : P)\ \rangle \\
& (\forall y, x \,|\, \langle x, y \rangle \in \sigma : \langle x, y \rangle \in \mu) \\
= & \langle\ (9.2) \text{ for multiple quantifications}\ \rangle \\
& (\forall y, x \,|\, : \langle x, y \rangle \in \sigma \Rightarrow \langle x, y \rangle \in \mu) \\
= & \langle\ (8.20)\ \rangle \\
& (\forall y \,|\, : (\forall x \,|\, : \langle x, y \rangle \in \sigma \Rightarrow \langle x, y \rangle \in \mu)) \\
\Rightarrow & \langle\ (9.27), (9.16), (9.12)\ \rangle \\
& (\forall y \,|\, : (\exists x \,|\, : \langle x, y \rangle \in \sigma) \Rightarrow (\exists x \,|\, : \langle x, y \rangle \in \mu)) \\
= & \langle\ (9.2)\ \rangle \\
& (\forall y \,|\, (\exists x \,|\, : \langle x, y \rangle \in \sigma) : (\exists x \,|\, : \langle x, y \rangle \in \mu)) \\
= & \langle\ (14.17)\ \rangle \\
& (\forall y \,|\, y \in \text{Ran}.\sigma : y \in \text{Ran}.\mu) \\
= & \langle\ (11.13),\ y \text{ d.n.o.f. in Ran}.\sigma, \text{Ran}.\mu\ \rangle \\
& \text{Ran}.\sigma \subseteq \text{Ran}.\mu\ .
\end{array}$$

2. (a) (4 points) Prove that

$$\vdash \imath_S \circ \imath_S = \imath_S \ .$$

**Note:** You may use YAUL, $\vdash \langle x, y\rangle \in \imath_S \ \equiv \ x \in S \ \wedge \ y = x$ without providing a proof.

**Answer:** Take $x, y, z$ to be fresh variables. By 14 RE and (9.16) for multiple quantifications, it suffices to prove that, $\langle x, y\rangle \in \imath_S \circ \imath_S \ \equiv \ \langle x, y\rangle \in \imath_S$ .

$$\begin{aligned}
& \langle x, y\rangle \in \imath_S \circ \imath_S \\
= \ & \langle \ (14.20),\ z \text{ d.n.o.f. in } \imath_S \ \rangle \\
& (\exists z \mid \ : \langle x, z\rangle \in \imath_S \ \wedge \ \langle z, y\rangle \in \imath_S) \\
= \ & \langle \ \text{YAUL} \ \rangle \\
& (\exists z \mid \ : x \in S \ \wedge \ z = x \ \wedge \ z \in S \ \wedge \ y = z) \\
= \ & \langle \ (9.19) \ \rangle \\
& (\exists z \mid z = x : x \in S \ \wedge \ z \in S \ \wedge \ y = z) \\
= \ & \langle \ (8.14) \ \rangle \\
& x \in S \ \wedge \ x \in S \ \wedge \ y = x \\
= \ & \langle \ (3.38) \ \rangle \\
& x \in S \ \wedge \ y = x \\
= \ & \langle \ \text{YAUL} \ \rangle \\
& \langle x, y\rangle \in \imath_S \ .
\end{aligned}$$

(b) (5 points) Prove that

$$\vdash \ (\forall n \mid n \geq 1 : (\imath_S)^n = \imath_S) \ .$$

**Answer:**

The base case is $n = 1$. This is $(\imath_S)^1 = \imath_S$ which follows from (14.25).

Assume $(\imath_S)^n = \imath_S$.

$$\begin{aligned}
& (\imath_S)^{n+1} \\
= \ & \langle \ (14.25) \ \rangle \\
& (\imath_S)^n \circ \imath_S \\
= \ & \langle \ \text{Assumption} \ \rangle \\
& \imath_S \circ \imath_S \\
= \ & \langle \ \text{Part (a)} \ \rangle \\
& \imath_S
\end{aligned}$$

3. (6 points) Let $S = \{1, 2, 3, 4\}$ and $\rho = \{\langle 3, 1\rangle, \langle 4, 4\rangle\}$ .

(a) Find the smallest relation containing $\rho$ which is reflexive on $S$ and symmetric.
**Answer:**
$\{\langle 3, 1\rangle, \langle 1, 3\rangle, \langle 4, 4\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle\}$

(b) Find the smallest relation containing $\rho$ which is symmetic and transitive.
**Answer:**
$\{\langle 3, 1\rangle, \langle 1, 3\rangle, \langle 4, 4\rangle, \langle 1, 1\rangle, \langle 3, 3\rangle\}$

(c) Find the smallest relation containing $\rho$ which is reflexive on $S$ and symmetric and transitive.
**Answer:**
$\{\langle 3, 1\rangle, \langle 1, 3\rangle, \langle 4, 4\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle\}$