York University

Faculty of Arts, Faculty of Science. Atkinson College

Math 2090

Midterm Test

SOLUTIONS

Instructions:

1. Time allowed : 75 minutes.
2. There are 6 questions on 7 pages. Page 3 is blank.
3. Answer all questions.
4. Prove, means prove using the proof methods and proof format of the text.
   If your reason is not a theorem on the list provided, or another question on this test, you must provide its proof.
5. Your reasons must indicate that you have verified any conditions concerning free occurrences of variables.

Question	**Points**	**Marks**
1	15	
2	25	
3	15	
4	20	
5	10	
6	15	
Total	100	

1. (15 points) Prove that,

$$\vdash T \subseteq S \quad \equiv \quad (S - T) \cup T \ = \ S \ .$$

Answer: Take z to be a fresh variable.

$$
\begin{array}{cl}
 & (S - T) \cup T \ = \ S \\
= & \langle \ (11.4),\ z \text{ d.n.o.f. in } (S - T) \cup T, S \ \rangle \\
 & (\forall z \,|\, : z \in (S - T) \cup T \ \equiv \ z \in S) \\
= & \langle \ (11.22),\ (11.20) \ \rangle \\
 & (\forall z \,|\, : (z \in S \wedge z \notin T) \vee z \in T \ \equiv \ z \in S) \\
= & \langle \ (3.44)(\text{b}) \ \rangle \\
 & (\forall z \,|\, : z \in S \vee z \in T \ \equiv \ z \in S) \\
= & \langle \ (3.57) \ \rangle \\
 & (\forall z \,|\, : z \in T \Rightarrow z \in S) \\
= & \langle \ (9.2) \ \rangle \\
 & (\forall z \,|\, z \in T : z \in S) \\
= & \langle \ (11.13) \ \rangle \\
 & T \subseteq S \ .
\end{array}
$$

2. (a) (10 points) Prove (11.57),

$$\vdash S = T \quad \equiv \quad S \subseteq T \ \wedge \ T \subseteq S \ .$$

Answer: Take z to be a fresh variable.

$$\begin{aligned}
& S = T \\
= \quad & \langle \ (11.4),\ z \text{ d.n.o.f. in } S, T \ \rangle \\
& (\forall z \,|\, : z \in S \ \equiv \ z \in T) \\
= \quad & \langle \ (3.80) \ \rangle \\
& (\forall z \,|\, : (z \in S \ \Rightarrow z \in T) \wedge (z \in T \ \Rightarrow z \in S)) \\
= \quad & \langle \ (8.15) \ \rangle \\
& (\forall z \,|\, : z \in S \ \Rightarrow z \in T) \wedge (\forall z \,|\, : z \in T \ \Rightarrow z \in S) \\
= \quad & \langle \ (9.2) \ \rangle \\
& (\forall z \,|\, z \in S : z \in T) \wedge (\forall z \,|\, z \in T : z \in S) \\
= \quad & \langle \ (11.13),\ z \text{ d.n.o.f. in } S, T \ \rangle \\
& S \subseteq T \ \wedge \ T \subseteq S \ .
\end{aligned}$$

(b) (15 points) Prove that,

$$\vdash \mathcal{P}S = \mathcal{P}T \quad \equiv \quad S = T \ .$$

Answer: Take z to be a fresh variable.

$$\begin{aligned}
& \mathcal{P}S = \mathcal{P}T \\
= \quad & \langle \ (11.4),\ z \text{ d.n.o.f. in } \mathcal{P}S, \mathcal{P}T \ \rangle \\
& (\forall z \,|\, : z \in \mathcal{P}S \ \equiv \ z \in \mathcal{P}T) \\
= \quad & \langle \ (11.23) \ \rangle \\
& (\forall z \,|\, : z \subseteq S \ \equiv \ z \subseteq T) \\
= \quad & \langle \ (9.13),\ (3.60) \text{ twice} \ \rangle \\
& (\forall z \,|\, : z \subseteq S \ \equiv \ z \subseteq T) \ \wedge \ (S \subseteq S \ \equiv \ S \subseteq T) \ \wedge \ (T \subseteq S \ \equiv \ T \subseteq T) \\
= \quad & \langle \ \text{Lemma: } \vdash S \subseteq S \ \rangle \\
& (\forall z \,|\, : z \subseteq S \ \equiv \ z \subseteq T) \ \wedge \ (\mathit{true} \ \equiv \ S \subseteq T) \ \wedge \ (T \subseteq S \ \equiv \ \mathit{true}) \\
= \quad & \langle \ (3.3) \ \rangle \\
& (\forall z \,|\, : z \subseteq S \ \equiv \ z \subseteq T) \ \wedge \ S \subseteq T \ \wedge \ T \subseteq S \\
= \quad & \langle \ \text{Part (a)} \ \rangle \\
& (\forall z \,|\, : z \subseteq S \ \equiv \ z \subseteq T) \ \wedge \ S = T \\
= \quad & \langle \ (3.84)(a), \text{ noting that for the contextual substitution, } z \text{ d.n.o.f. in } S, T \ \rangle \\
& (\forall z \,|\, : z \subseteq S \ \equiv \ z \subseteq S) \ \wedge \ S = T \\
= \quad & \langle \ (3.3) \ \rangle
\end{aligned}$$

$$
\begin{array}{ll}
 & (\forall z \mid\ : true) \ \wedge\ S = T \\
= & \langle\ (9.8)\ \rangle \\
 & true \ \wedge\ S = T \\
= & \langle\ (3.39)\ \rangle \\
 & S = T\ .
\end{array}
$$

Proof of Lemma:
Take z to be a fresh variable.

$$
\begin{array}{ll}
 & S \subseteq S \\
= & \langle\ (11.13),\ z \text{ d.n.o.f. in } S\ \rangle \\
 & (\forall z \mid z \in S : z \in S) \\
= & \langle\ (9.2)\ \rangle \\
 & (\forall z \mid\ : z \in S \Rightarrow z \in S) \\
= & \langle\ (3.71)\ \rangle \\
 & (\forall z \mid\ : true) \\
= & \langle\ (9.8)\ \rangle \\
 & true\ .
\end{array}
$$

3. (15 points) Consider the following attempt to prove $\{y\} = \{x, y\}$.
If a proof step is correct, provide complete justification. If the justification uses a Lemma, state it, then provide a proof.
If a proof step is incorrect, explain why it is incorrect.

$$
\begin{array}{ll}
 & \{y\} = \{x, y\} \\
= & \left\langle \begin{array}{l} \text{This step is incorrect.} \\ \text{Observe that } y \text{ occurs free in } \{y\} \text{ and in } \{x,y\}\text{. One cannot use (11.4)} \\ \text{if the variable of quantification occurs free in either of } \{y\} \text{ and } \{x,y\}. \end{array} \right\rangle \\
 & (\forall y \mid : y \in \{y\} \ \equiv \ y \in \{x, y\}) \\
= & \left\langle \begin{array}{l} \text{Lemma: } \vdash y \in \{y\}. \\ \text{Lemma: } \vdash y \in \{x, y\}. \end{array} \right\rangle \\
 & (\forall y \mid : \mathit{true} \ \equiv \ \mathit{true}) \\
= & \langle \text{ (3.3) or alternatively, (1.2) } \rangle \\
 & (\forall y \mid : \mathit{true}) \\
= & \langle \text{ (9.8) } \rangle \\
 & \mathit{true} \ .
\end{array}
$$

Proof of Lemma: $\vdash y \in \{y\}$. Take z to be a fresh variable.

$$
\begin{array}{ll}
 & y \in \{y\} \\
= & \langle \text{ Abbreviation, } z \text{ d.n.o.f. in } y \ \rangle \\
 & y \in \{z \mid z = y : z\} \\
= & \langle \text{ (11.3), } z \text{ d.n.o.f. in } y \ \rangle \\
 & (\exists z \mid z = y : y = z) \\
= & \langle \text{ (8.14), } z \text{ d.n.o.f. in } y \ \rangle \\
 & y = y \ .
\end{array}
$$

Proof of Lemma: $\vdash y \in \{x, y\}$. Take z to be a fresh variable.

$$
\begin{array}{ll}
 & y \in \{x, y\} \\
= & \langle \text{ Abbreviation, } z \text{ d.n.o.f. in } x, y \ \rangle \\
 & y \in \{z \mid z = x \lor z = y : z\} \\
= & \langle \text{ (11.3), } z \text{ d.n.o.f. in } y \ \rangle \\
 & (\exists z \mid z = x \lor z = y : y = z) \\
= & \langle \text{ (8.18) } \rangle \\
 & (\exists z \mid z = x : y = z) \ \lor \ (\exists z \mid z = y : y = z) \\
= & \langle \text{ (8.14) } \rangle \\
 & (\exists z \mid z = x : y = z) \ \lor \ y = y \\
= & \langle \text{ (1.2) } \rangle \\
 & (\exists z \mid z = x : y = z) \ \lor \ \mathit{true} \\
= & \langle \text{ (3.29) } \rangle \\
 & \mathit{true} \ .
\end{array}
$$

4. (a) (10 points) Prove that,

$$\vdash \{u\} = \{v, w\} \;\Rightarrow\; u = v \wedge u = w \;.$$

Answer: Take z to be a fresh variable.

$$\begin{array}{cl}
 & \{u\} = \{v, w\} \\
= & \langle\ (11.4),\ z \text{ d.n.o.f. in } \{u\}, \{v, w\}\ \rangle \\
 & (\forall z \mid\ : z \in \{u\} \equiv z \in \{v, w\}) \\
= & \langle\ \text{Abbreviation},\ z \text{ d.n.o.f. in } u, v, w\ \rangle \\
 & (\forall z \mid\ : z \in \{z \mid z = u\} \equiv z \in \{z \mid z = v \vee z = w\}) \\
= & \langle\ (11.7)\ \rangle \\
 & (\forall z \mid\ : z = u \equiv z = v \vee z = w) \\
= & \langle\ (9.13),\ (3.60) \text{ twice}\ \rangle \\
 & (\forall z \mid\ : z = u \equiv z = v \vee z = w) \wedge (v = u \equiv v = v \vee v = w) \wedge (w = u \equiv w = v \vee w = w) \\
\Rightarrow & \langle\ (3.76)(\text{b})\ \rangle \\
 & (v = u \equiv v = v \vee v = w) \wedge (w = u \equiv w = v \vee w = w) \\
= & \langle\ (1.2)\ \rangle \\
 & (v = u \equiv \mathit{true} \vee v = w) \wedge (w = u \equiv w = v \vee \mathit{true}) \\
= & \langle\ (3.29)\ \rangle \\
 & (v = u \equiv \mathit{true}) \wedge (w = u \equiv \mathit{true}) \\
= & \langle\ (3.2)\ \rangle \\
 & v = u \wedge w = u \\
= & \langle\ (1.3)\ \rangle \\
 & u = v \wedge u = w
\end{array}$$

(b) (10 points) Prove that,

$$\vdash \{\{x\}\} = \{\{y\}, \{y, z\}\} \;\Rightarrow\; x = y \wedge y = z \;.$$

Answer:

$$\begin{array}{cl}
 & \{\{x\}\} = \{\{y\}, \{y, z\}\} \\
\Rightarrow & \langle\ \text{Part (a)}\ \rangle \\
 & \{x\} = \{y\} \wedge \{x\} = \{y, z\} \\
\Rightarrow & \langle\ (3.76)(\text{b})\ \rangle \\
 & \{x\} = \{y, z\} \\
\Rightarrow & \langle\ \text{Part (a)}\ \rangle \\
 & x = y \wedge x = z \\
= & \langle\ (3.84)(\text{a})\ \rangle \\
 & x = y \wedge y = z \;.
\end{array}$$

5. (10 points) Is $\#\{u, v\} = \#\{x, y\}$ a theorem? If it is a theorem, provide a proof. If it is not a theorem, carefully explain why not.

Answer: It is not a theorem. Take $\{0, 1, 2\}$ as Universe of Discourse. Assign 0 to the free occurrence of u, 0 to the free occurrence of v, 1 to the free occurrence of x, and 2 to the free occurrence of y. Then $\#\{u, v\}$ evaluates to 1 while $\#\{x, y\}$ evaluates to 2 and $1 \neq 2$.

6. (15 points) Use Mathematical Induction to prove that,

$$\vdash (\forall n : \mathbb{N} \mid \ : (+k : \mathbb{N} \mid 0 \le k \le n : 2k + 1) = (n + 1)^2)\ .$$

Clearly state the Mathematical Induction Inference Rule which you use.
Answer: To prove $\vdash (\forall n \mid \ : P.n)$ it suffices to prove $\vdash P.0$ and $\vdash P.n \Rightarrow P.n + 1$.

Proof of $\vdash P.0$:

$$\begin{array}{ll} & (+k \mid 0 \le k \le 0 : 2k + 1) \\ = & \langle \text{ Arithmetic, } \vdash 0 \le k \le 0 \equiv k = 0 \ \rangle \\ & (+k \mid k = 0 : 2k + 1) \\ = & \langle \ (8.14) \ \rangle \\ & 2 \cdot 0 + 1 \\ = & \langle \text{ Arithmetic } \rangle \\ & (0 + 1)^2\ . \end{array}$$

Proof of $\vdash P.n \Rightarrow P.n + 1$:
Assume $(+k \mid 0 \le k \le n : 2k + 1) = (n + 1)^2$.

$$\begin{array}{ll} & (+k \mid 0 \le k \le n + 1 : 2k + 1) \\ = & \langle \text{ Arithmetic, } \vdash 0 \le k \le n + 1 \equiv 0 \le k \le n \vee k = n + 1 \ \rangle \\ & (+k \mid 0 \le k \le n \vee k = n + 1 : 2k + 1) \\ = & \langle \text{ (8.16), Arithmetic, } \vdash 0 \le k \le n \wedge k = n + 1 \equiv \mathit{false} \ \rangle \\ & (+k \mid 0 \le k \le n : 2k + 1) + (+k \mid k = n + 1 : 2k + 1) \\ = & \langle \text{ Assumption, (8.14), } k \text{ d.n.o.f. in } n + 1 \ \rangle \\ & (n + 1)^2 + (2(n + 1) + 1) \\ = & \langle \text{ Algebra } \rangle \\ & ((n + 1) + 1)^2\ . \end{array}$$

The end