Gauss’s Lemma and the Irrationality of Roots

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Browsing through the Biscuits [1], I came across an article by Harley Flanders that was originally published in this Magazine [3, 1999]. In it, Flanders rephrases T. Estermann’s [2, 1975] proof of the irrationality of $\sqrt{2}$ to show the irrationality of $\sqrt{k}$ for every non-square positive integer $k$. It turns out that Estermann’s idea can be further extended to conclude that $\sqrt[k]{n}$ is either an integer or irrational, for every pair $k, n$ of positive integers. Even more is true: Estermann’s idea leads to a simple proof of Gauss’s Lemma:

**Theorem 1. (Gauss’s Lemma)** Every real root of a monic polynomial with integer coefficients is either an integer or irrational.

For the case of $\sqrt[k]{n}$, apply the theorem to the monic polynomial $P(x) = x^n - k$.

**The Standard Proof**

The standard proof of Gauss’s lemma (e.g. [4, p. 41]) runs this way. Let $r$ be a real root of the monic polynomial

$$P(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0,$$

where $n$ is a positive integer and $c_0, \ldots, c_{n-1}$ are integers. If $r$ is not irrational, then represent it as a fraction $a/b$ with $(a, b) = 1$. Multiply the equality $P(a/b) = 0$ by $b^n$ and isolate the leading term to obtain

$$a^n = -\left( c_{n-1}a^{n-1}b + \cdots + c_0b^n \right).$$

The right side is a multiple of $b$, but if $b > 1$ then the left side is not. Therefore $b = 1$ and $r$ is an integer.

Though quite simple and short, this proof hinges in an essential way on divisibility properties of the integers. One must know that if $b$ has a prime factor $p$ that does not divide $a$, then $p$ does not divide any positive power of $a$. The use of this principle in some form seems unavoidable.

An alternative proof is offered below. It uses neither prime factorization nor divisibility; it does not even require one to know what it means for a number to be prime. It assumes only that the set of natural numbers is well ordered, i.e. that each of its nonempty subsets has a least element.
The Alternative Proof

Let $r$ be a real root of the monic polynomial (1), and suppose that $r$ is rational but not an integer. Then $r$ is uniquely (and strictly) sandwiched between two consecutive integers, say $q < r < q + 1$.

Since $r$ is rational, so are its first $(n-1)$ powers, $r^1, r^2, \ldots, r^{n-1}$. Consequently, the set

$$M := \{m > 0 \mid m, mr, mr^2, \ldots, mr^{n-1} \text{ are integers}\}$$

is nonempty. Since $r$ is a root of the polynomial, $r^n = -(c_{n-1}r^{n-1} + \cdots + c_0)$. For every $m \in M$, $m(c_{n-1}r^{n-1} + \cdots + c_1 r + c_0)$ is an integer, and thus $mr^n$ is an integer as well.

We claim that, for every $m \in M$ there is an $m' \in M$ with $0 < m' < m$. Given $m \in M$, set $m' = (r - q)m$. Then for each $i = 0, \ldots, n-1$ we have $mr^i = mr^{i+1} - qmr^i$, which is an integer; so $m' \in M$; and $0 < m' < m$ because $0 < r - q < 1$.

Thus $M$ cannot have a least element. It follows that $r$ must be an integer or irrational.

Should one be interested in $\sqrt{k}$ only, specializing the above argument to polynomials of form $x^n - k$ yields a direct proof. Further specialization to the single polynomial $x^n - 2$ leads all the way back to Estermann’s argument [2] for the irrationality of $\sqrt{2}$.

The simple alternative argument presented here deserves to be widely publicized and should become a standard textbook proof.

Acknowledgment

I have known this proof in the case of $\sqrt{k}$ for many years and have used it in teaching and informal chats with colleagues and students, but never considered writing it down for submission to a journal until being recently encouraged to do so by Arthur Benjamin. Subsequently, I was challenged by another colleague to extend the idea to prove the stronger result. I wish to thank them both, each for his respective contribution, without which this note would not materialize.

REFERENCES

1. A. T. Benjamin & E. Brown (Editors), Biscuits of Number Theory, MAA 2009.

Summary

An idea of T. Estermann (1975) for demonstrating the irrationality of $\sqrt{2}$ is extended to obtain a conceptually simple proof of Gauss’s Lemma, according to which real roots of monic polynomials with integer coefficients are either integers or irrational. The standard proof of the lemma is also reviewed.