Math 3110 Homework 1 due September 21 at Noon
SOLUTIONS

- Problem 4, page 11:

  If $x \in (A \setminus B) \cup (B \setminus A)$ and $x \in (A \setminus B)$, then $x \in A$ which gives $x \in A \cup B$. Since $x \in A$ but $x \notin B$, it follows that $x \notin A \cap B$. Then $x \in (A \cup B) \setminus (A \cap B)$. Otherwise, $x \in (B \setminus A)$, and $x \in B$ which gives $x \in A \cup B$. Since $x \in B$ but $x \notin A$, and $x \notin A \cap B$. Then $x \in (A \cup B) \setminus (A \cap B)$. This proves $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$.

  If $x \in (A \cup B) \setminus (A \cap B)$ then $x \in A \cup B$ from which $x \in A$ or $x \in B$. If $x \in A$ as $x \notin A \cap B$, we must have $x \notin B$, from which $x \in A \setminus B$. Otherwise $x \in B$, and as $x \notin A \cap B$, we must have $x \notin A$, from which $x \in B \setminus A$. Then $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$ and equality follows.

- Problem 12, page 11:

  If $y \in f(E \cup F)$ there exists $x \in E \cup F$, $f(x) = y$. If $x \in E$ then $y \in f(E)$. Otherwise $x \in F$ and $y \in f(F)$. Then $y \in f(E) \cup f(F)$, from which $f(E \cup F) \subseteq f(E) \cup f(F)$.

  If $y \in f(E) \cup f(F)$ and $y \in f(E)$, then there exists $x \in E$ such that $f(x) = y$. As if $x \in E$ it follows that $x \in E \cup F$, which gives $y = f(x) \in f(E \cup F)$. Otherwise, if $y \in f(E) \cup f(F)$ and $y \in f(F)$, then there exists $x \in F$ such that $f(x) = y$. As from $x \in F$ it follows that $x \in E \cup F$, from which $y = f(x) \in f(E \cup F)$. Then $f(E) \cup f(F) \subseteq f(E \cup F)$ and equality follows.

  If $y \in f(E \cap F)$ there exists $x \in E \cap F$, $f(x) = y$. Since $x \in E \cap F$, $x \in E$ from which $f(x) = y \in f(E)$. Since $x \in E \cap F$, $x \in F$ from which $f(x) = y \in f(F)$. Then $f(E \cap F) \subseteq f(E) \cap f(F)$.

- Problem 13, page 11:

  If $x \in f^{-1}(G \cup H)$ then $f(x) \in G \cup H$. If $f(x) \in G$ then $x \in f^{-1}(G)$. Otherwise, $f(x) \in H$ and $x \in f^{-1}(H)$. Then $x \in f^{-1}(G) \cup f^{-1}(H)$ giving $f^{-1}(G \cup H) \subseteq x \in f^{-1}(G) \cup f^{-1}(H)$.

  If $x \in f^{-1}(G) \cup f^{-1}(H)$ and $x \in f^{-1}(G)$ then $f(x) \in G$ and $f(x) \in G \cup H$ so that $x \in f^{-1}(G \cup H)$. Otherwise, $f(x) \in H$ from which $f(x) \in G \cup H$ so that $x \in f^{-1}(G \cup H)$. This gives $x \in f^{-1}(G \cup H)$ hence $f^{-1}(G) \cup f^{-1}(H) \subseteq f^{-1}(G \cup H)$ and equality follows.

If $x \in f^{-1}(G \cap H)$ then $f(x) \in G \cap H$. As $f(x) \in G$, $x \in f^{-1}(G)$ and as $f(x) \in H$, $x \in f^{-1}(H)$. Then $x \in f^{-1}(G) \cap f^{-1}(H)$ from which $f^{-1}(G \cap H) \subseteq f^{-1}(G) \cap f^{-1}(H)$.

If $x \in f^{-1}(G) \cap f^{-1}(H)$ then $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$. As $x \in f^{-1}(G)$, $f(x) \in G$ and as $x \in f^{-1}(H)$, $f(x) \in H$. Then $f(x) \in (G \cap H)$ which gives $x \in f^{-1}(G \cap H)$. Then $f^{-1}(G) \cap f^{-1}(H) \subseteq f^{-1}(G \cap H)$ and equality follows.

- Problem 15, page 11:

  The simplest way to find a solution is to first sketch the graph of the desired function. For the line through (a,0) and (b,1),

$$f(x) = \frac{x}{b-a} - \frac{a}{b-a} \; .$$

- Problem 16, page 11:

  Take $f : \mathbb{R} \to \mathbb{R}$ to be given by $f(x) = x$.
  Take $g : \mathbb{R} \to \mathbb{R}$ to be given by $g(x) = 1$.
  Then $(g \circ f)(x) = 1$ and $(f \circ g)(x) = 1$.

- Problem 17, page 11:

  (a): If $x \in E$, then $f(x) \in f(E)$ from which $x \in f^{-1}(f(E))$. In general, $E \subseteq f^{-1}(f(E))$.

  If $x \in f^{-1}(f(E))$, $f(x) \in f(E)$ so that there exists $y \in E$, $f(x) = f(y)$. As f is injective, $x = y$ from which $x \in E$. Then $f^{-1}(f(E)) \subseteq E$ and equality follows.

  Let $f : (-1,1) \to \mathbb{R}$, with $f(x) = x^2$ and E be $(-1,0]$. Then $f(E) = [0,1)$, $f^{-1}(f(E)) = (-1,1) \neq E$.

  (b): If $y \in f(f^{-1}(H))$ then we can find $x \in f^{-1}(H)$ such that $f(x) = y$. As $x \in f^{-1}(H)$, $f(x) \in H$ from which $y \in H$. Then $f(f^{-1}(H)) \subseteq H$.

  If $y \in H$, as f is surjective, there exists $x \in A$ such that $f(x) = y$. As $y \in H$, $x \in f^{-1}(H)$ and $y \in f(f^{-1}(H))$. Then $H \subseteq f(f^{-1}(H))$ and equality follows.

- Problem 14, page 11:

  We have to show that $f : \mathbb{R} \to (-1,1)$ is injective and surjective.

  Assume $f(x) = f(y)$. Squaring and simplifying $\frac{x}{\sqrt{x^2+1}} = \frac{y}{\sqrt{y^2+1}}$ one obtainss $x^2 = y^2$ from which $x = \pm y$. As $\sqrt{x^2+1} > 0$, x and $f(x)$

always have the same sign. In particular, as $f(x)$ and $f(y)$ have the same sign, so do x and y which gives $x = y$.

Let $b \in (-1, 1)$. We must show that there exists x for which $\frac{x}{\sqrt{x^2+1}} = b$.

Solving for x gives $x = \frac{b}{\sqrt{1-b^2}}$.