Math 3110 Homework 2 due October 5 at Noon
SOLUTIONS

1. Problem 11, page 16:

   The sum of the first n positive odd numbers is n^2.

   For $n = 1$ we obtain $1 = 1^2$ which clearly holds.

   Assume that for some k, $1 + 3 + \cdots + (2k - 1) = k^2$.
   Then $1 + 3 + \cdots + (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2$.

2. Problem 18, page 16:

   For $n = 1$ we obtain $\frac{1}{\sqrt{1}} \geq \sqrt{1}$ which clearly holds.

   Assume that for some k, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} \geq \sqrt{k}$.
   Then $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k} + \frac{1}{\sqrt{k+1}}$.
   It suffices to prove $\sqrt{k} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$. This holds provided $\sqrt{k}\sqrt{k+1} + 1 \geq k + 1$ which follows since $\sqrt{k}\sqrt{k+1} \geq \sqrt{k}\sqrt{k} = k$.

3. Problem 9, page 21:

   This can be done from scratch by following the proof of 1.3.12. Alternatively, apply 1.3.12 with $A_1 = S, A_m = T$ for $m \geq 2$, to obtain that $S \cup T$ is countable. But $S \subseteq S \cup T$ and S is not finite. Then $S \cup T$ is not finite from which $S \cup T$ is denumerable.

4. Problem 12, page 21:

   Since a countable union of countable sets is countable, it suffices to show that for each n, the number of subsets of $\mathbb{N}$ with n elements is countable. Let S be a suset of $\mathbb{N}$ with n elements. Write $S = \{i_1, i_2, \ldots, i_n\}$ where the elements i_k are ordered so that $i_1 < i_2 < \cdots < i_n$. Then $\{\text{subsets of } \mathbb{N} \text{ with } n \text{ elements}\} \to \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ given by $S \mapsto (i_1, i_2, \ldots, i_n)$ is an injective map into a countable set. Then the set of subsets of $\mathbb{N}$ with n elements is countable.

5. Problem 4, page 29:

$$\begin{aligned} & a \cdot a = a \\ \Leftrightarrow\quad & a \cdot a + (-a) = 0 \\ \Leftrightarrow\quad & a \cdot a + (-1) \cdot a = 0 \\ \Leftrightarrow\quad & a \cdot (a + (-1)) = 0 \\ \Leftrightarrow\quad & a = 0 \ \text{ or } \ a + (-1) = 0 \\ \Leftrightarrow\quad & a = 0 \ \text{ or } \ a = 1 \ . \end{aligned}$$

6. Problem 7, page 29:

   Assume that there exist integers p, q without common factors so that $\left(\frac{p}{q}\right)^2 = 3$. Then $p^2 = 3q^2$. As $3 \,|\, p^2$ and 3 is prime, $3 \,|\, p$. Write $p = 3p'$. Then $p(p')^2 = 3q^2$, from which

$3(p')^2 = q^2$. As above, $3 \mid q^2$ from which $3 \mid q$. Then 3 is a common factor of p and q which contradicts the assumption.

7. Problem 8, page 29:

   (a): Write $x = \dfrac{p}{q}$ and $y = \dfrac{r}{s}$ where $p, q, r, s \in \mathbb{Z}$ and $q \neq 0, s \neq 0$. We have $xy = \dfrac{pr}{qs}$ with $pr \in \mathbb{Z}, qs \in \mathbb{Z}, qs \neq 0$. Similarly, $x + y = \dfrac{ps + rq}{qs}$ where $ps + rq \in \mathbb{Z}, qs \in \mathbb{Z}, qs \neq 0$.

   (b): If $x + y$ is rational and x is rational, $y = (x + y) + (-1)x$ is rational by (a).

   If xy is rational and x is rational, $x \neq 0$, then $\left(\dfrac{1}{x}\right)$ is rational, and $y = \left(\dfrac{1}{x}\right)(xy)$ is rational by (a).

8. Problem 14, page 29:

   We are given that $0 \le a < b$. As $a < b$ and $a \ge 0$, we have $a^2 \le ab$. As $a < b$ and $b > 0$, we have $ab < b^2$. Put these together to obtain $a^2 \le ab < b^2$.

   Take $a = 0$ and $b = 1$. It is not the case that $0 < 0 < 1$.

9. Problem 16, page 29:

   (b): We prove that if $x^2 > 1$ then $x < -1$ or $x > 1$ and if $x^2 < 4$ then $-2 < x < 2$. Together they give $-2 < x < -1$ or $1 < x < 2$.

$$
\begin{aligned}
& 1 < x^2 \\
\Leftrightarrow\quad & x^2 - 1 > 0 \\
\Leftrightarrow\quad & (x-1)(x+1) > 0 \\
\Leftrightarrow\quad & \{x - 1 < 0 \ \text{ and } \ x + 1 < 0\} \quad \text{or} \quad \{x - 1 > 0 \ \text{ and } \ x + 1 > 0\} \\
\Leftrightarrow\quad & x < -1 \ \text{ or } \ x > 1
\end{aligned}
$$

and

$$
\begin{aligned}
& x^2 < 4 \\
\Leftrightarrow\quad & x^2 - 4 < 0 \\
\Leftrightarrow\quad & (x-2)(x+2) < 0 \\
\Leftrightarrow\quad & x - 2 < 0 \ \text{ and } \ x + 2 > 0 \\
\Leftrightarrow\quad & x < 2 \ \text{ and } \\
\Leftrightarrow\quad & -2 < x < 2 \ .
\end{aligned}
$$

   (b):

$$
\begin{aligned}
& \frac{1}{x} < x^2 \\
\Leftrightarrow\quad & x^2 - \frac{1}{x} > 0 \\
\Leftrightarrow\quad & \frac{x^3 - 1}{x} > 0 \\
\Leftrightarrow\quad & \frac{x-1}{x}(x^2 + x + 1) > 0
\end{aligned}
$$

$$\Leftrightarrow \quad \frac{x-1}{x} > 0 \quad \text{(as } x^2+x+1 \text{ being a sum of squares is never negative)}$$
$$\Leftrightarrow \quad \{x-1<0 \text{ and } x<0\} \quad \text{or} \quad \{x-1>0 \text{ and } x>0\}$$
$$\Leftrightarrow \quad x<0 \text{ or } x>1 \, .$$

10. Problem 18, page 29:

    Assume $a - b \leq \epsilon$ all $\epsilon > 0$.

    If $a \geq b$ then $0 \leq a - b < \epsilon$ for all $\epsilon > 0$ from which $a = b$.

    Otherwise $a < b$. Together these give $a \leq b$.

11. Problem 22, page 29:

    (a): We have $c^1 = c$.

    The proof is by induction starting at $n_0 = 2$. Observe that as $c > 1$ we have $c > 0$. Multiply to obtain $c \cdot c > 1 \cdot c$, i.e., $c^2 > c$.

    Now assume that for some k, $c^k > c$. Then as $c > 0$, we have $c \cdot c^k > c \cdot c > c$, i.e., $c^{k+1} > c$.

    (b): Observe that as $0 < c < 1$ it follows that $\frac{1}{c} > 1$.

    By (a), we have $\left(\frac{1}{c}\right)^n > \frac{1}{c}$ from which $\frac{1}{c^n} > \frac{1}{c}$. Multiplying both sides by the positive quantity c^{n+1} gives $c > c^n$.