Math 3110 Homework 3 due November 2 at Noon
SOLUTIONS

1. Problem 15, page 34:
   Choose δ and ϵ such that $\delta + \epsilon < |b-a|$. Consider $x \in V_\delta(a) \cap V_\epsilon(b)$.
   Then $|b-a| = |(x-a)+(b-x)| \leq |x-a| + |b-x| < \delta + \epsilon < |b-a|$, a contradiction, from which, $V_\delta(a) \cap V_\epsilon(b) = \emptyset$.

2. Problem 1, page 38:
   –1 is a lower bound.
   If x were an upper bound, then in particular, $x \geq x+1$, a contradiction.
   To show that 0 is the greatest lower bound, assume to the contrary that the greatest lower bound is some number $\epsilon < 0$. Then as $\frac{\epsilon}{2} < 0$, $\frac{\epsilon}{2}$ is also a lower bound and we have $\epsilon \leq \frac{\epsilon}{2}$, a contradiction.

3. Problem 3, page 38:
   For $n > 1$, $\frac{1}{1} > \frac{1}{n}$ so that 1 is an upper bound. Since $1 \in S_3$, 1 must be the least uppper bound. If $x < 0$ is the greatest lower bound then consider $\frac{x}{2} < 0$. $x < \frac{x}{2} < 0$ and $\frac{x}{2}$ would be a lower bound, a contradition. The $\operatorname{glb} S_3 \geq 0$.

4. Problem 6, page 38:
   Let $s \in S$ be an upper bound for S. For any $\epsilon \geq 0$, we have $s > s - \epsilon$ from which it follows that s is the least upper bound.

5. Problem 8, page 38:
   From the definition of least upper bound, if u is the least upper bound, then $u - \epsilon$, in this case $u - \frac{1}{n}$ is not an upper bound. If u is the least upper bound then u is in any case an upper bound. Since $\frac{1}{n} > 0$, we have $u + \frac{1}{n} > u$ so that $u + \frac{1}{n}$ is an upper bound as well.

6. Problem 10, page 38:
   Any upper bound for S (including of course $\operatorname{lub}(S)$) must be an upper bound of the subset S_0. Then $\operatorname{lub}(S) \geq \operatorname{lub}(S_0)$. Similarly, $\operatorname{glb}(S) \leq \operatorname{glb}(S_0)$. As $S_0 \neq \emptyset$ take $s \in S_0$. Then $\operatorname{glb}(S_0) \leq s$ and $\operatorname{lub}(S_0) \geq s$ so that by transitivity, $\operatorname{glb}(S_0) \leq \operatorname{lub}(S_0)$.

7. Problem 3, page 43:
   We must show u is an upper bound and u is the least upper bound.
   Consider $u - \epsilon$. By the Archimedian property we can find n such that $\frac{1}{n} < \epsilon$. As $1 - \frac{1}{n}$ is not an upper bound for S, and $1 - \frac{1}{n} > 1 - \epsilon$, $1 - \epsilon$ cannot be an upper bound for S either.
   To prove u is an upper bound, assume the contrary. Let $v \in S$, $v > u$. By the Archimedian property, we can find n such that $u < u + \frac{1}{n} < v$. This contradicts that $u + \frac{1}{n}$ is an upper bound for S.

8. Problem 6, page 43:
   Consider $x = a + b \in A + B$. As $a \leq \sup A$ and $b \leq \sup B$, we have $x = a + b \leq \sup A + \sup B$. Consider $\sup A + \sup B - \epsilon = (\sup A - \frac{\epsilon}{2}) + (\sup B - \frac{\epsilon}{2})$. There exists $a \in A$ such that $a > \sup A - \frac{\epsilon}{2}$ and there exists $b \in B$ such that $b > \sup B - \frac{\epsilon}{2}$. Then $a + b > \sup A + \sup B - \epsilon$, from which $\sup A + \sup B$ must be the least upper bound. The argument for the inf follows the same pattern.

9. Problem 7, page 43:
For equality just take $f = g$. For strict inequality, take $X = \{0, 1\}$. Define $f(0) = 0,\ f(1) = 1,\ g(0) = 1,\ g(1) = 0$.

10. Problem 12, page 43:
Assume that we have $m - 1 \leq x < m$ and $n - 1 \leq x < n$. Then as $-n < -x \leq 1 - n$ we have $m - n - 1 < 0 < m - n + 1$ from which $-1 < n - m < 1$. As $n - m$ is an integer we must have $m - n = 0$.

11. Problem 18, page 43:
Since $u > 0$ the condition is the same as the existence of r, $\frac{x}{u} < r < \frac{y}{u}$. This is 2.4.8.

12. Problem 2, page 50:
Let a be any lower bound, b be any upper bound. Then $S \subseteq [a.b]$. Conversely, if $S \subseteq [c, d]$, then we have $c \leq s$ for all $s \in S$ and $d \geq s$ for all $s \in S$.

13. Problem 3, page 50:
It follows from problem 2 that $S \subseteq [\inf S, \sup S]$. Now if $S \subseteq I = [a, b]$, then a is a lower bound and b an upper bound for S. Then $a \leq \inf S$ and $b \geq \sup S$, from which $[\inf S, \sup S] \subseteq [a, b]$.

14. Problem 7, page 50:
Any element x of the intersection must satisfy $x \geq 0$ and $x \leq \frac{1}{n}$ for all n. By the Archimedian property, the only possibility is $x = 0$.

15. Problem 8, page 50:
See problem 7. As $\bigcap J_n \subseteq \bigcap I_n$ the only possible element of $\bigcap J_n$ is 0. But 0 is not in J_n for any n.

16. Problem 9, page 50:
If $x \in \bigcap K_n$ then $x > n$ for all $n \in \mathbb{N}$. But the Archimedian property, $\mathbb{N}$ is not bounded above.