Math 3110 Homework 4 due November 16 at Noon
SOLUTIONS

1. Problem 8, page 60:
For $x_n \to 0$ the condition is, given $\epsilon > 0$ there exists N such that $n > N \;\Rightarrow\; |x_n - 0| < 0$, i.e., $|x_n| < 0$.
$|x_n| \to 0$ the condition is, given $\epsilon > 0$ there exists N such that $n > N \;\Rightarrow\; ||x_n| - 0| < 0$, i.e., $|x_n| < 0$.
These conditions are identical.
If $x_n = (-1)^n$, we have $|x_n| \to 1$ but (x_n) does not converge.

2. Problem 9, page 60:
Choose n sufficiently large that $0 < x_n < \epsilon^2$ for $n > N$. Then $0 < \sqrt{x_n} < \epsilon$.

3. Problem 10, page 60:
Choose $\epsilon = \frac{x}{2}$. There exists M such that $n > M \;\Rightarrow\; |x_n - x| < \frac{x}{2}$, i.e., $\frac{x}{2} < x_n < \frac{3x}{2}$. Take $n > M$. As $x_n > \frac{x}{2}$, it follows that $x_n > 0$.

4. Problem 14, page 60:
By **3.1.1 (d)** we have $n^{\frac{1}{n}} \to 1$ from which $(2n)^{\frac{1}{2n}} \to 1$. Squaring gives $(2n)^{\frac{1}{n}} \to 1$.

5. Problem 15, page 60:
Write $\frac{n^2}{n!} = \frac{n}{n-1} \cdot \frac{1}{(n-2)!}$ and observe that for $n > 2$, we have $\frac{n}{n-1} < 2$. then $\frac{n^2}{n!} < \frac{2}{(n-2)!} < \frac{2}{n-2}$.
Let $\epsilon > 0$. By choosing n sufficiently large we can ensure $\frac{2}{n-2} < \epsilon$.

6. Problem 16, page 60:
Using the hint given, as $2(\frac{2}{3})^{n-2} \to 0$, we have $\frac{2^n}{n} \to 0$. The hint can be proved using Mathematical Induction. For $n = 3$ the result holds by direct observation. Assume $k \geq 3$ and $\frac{2^k}{k!} \leq 2\left(\frac{2}{3}\right)^{k-2}$. Then $\frac{2^{k+1}}{(k+1)!} = \frac{2}{k+1}\frac{2^k}{k!} < \frac{2}{3}\frac{2^k}{k!} \leq \frac{2}{3} \cdot 2\left(\frac{2}{3}\right)^{k-2}$.

7. Problem 2, page 67:
For (a), take $X = (n)$ amd $Y = (-n)$.
For (b), take $x_n = n$ for n even, 1 for n odd and $y_n = \frac{1}{n}$ for n even, 1 for n odd.
Note that if we take $X = ((-1)^n)$ and $Y = ((-1)^{n+1})$, the conditions for both (a) and (b) are satisfied.

8. Problem 3, page 67:
Y is $(X+Y) - X$. The standard $\frac{\epsilon}{2}$ argument for the difference of two convergent series gives Y is convergent.

9. Problem 6, page 67:
For (a), as $2 + \frac{1}{n} \to 2$ we have $(2 + \frac{1}{n})^2 \to 4$.
For (b), as $\left|\frac{(-1)^n}{n+2}\right| \to 0$ we have $\frac{(-1)^n}{n+2} \to 0$.
For (c), we have $\frac{\sqrt{n}-1}{\sqrt{n}+1} \cdot \frac{\sqrt{n}-1}{\sqrt{n}-1} = \frac{n-2\sqrt{n}+1}{n-1} \to 1$.
For (d) as $\frac{n+1}{n} \to 1$ and $\frac{1}{\sqrt{n}} \to 0$ we have $\frac{n+1}{n\sqrt{n}} \to 1 \cdot 0 = 0$

10. Problem 9, page 67:
We have $y_n = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}}$. As $\frac{1}{2\sqrt{n}} \to 0$ we have $y_n \to 0$.
The same calculation gives $\sqrt{n}y_n = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \to \frac{1}{2}$.

11. Problem 14, page 67:
Rewrite $z_n = ((\frac{a^n}{b^n}+1)b^n)^{\frac{1}{n}}$. As $(b^n)^{\frac{1}{n}} = b$ it suffices to prove $(\frac{a^n}{b^n}+1)^{\frac{1}{n}} \to 1$. Set $r = \frac{a}{b}$. Then $0 < r < 1$. Write $(r^n+1)^{\frac{1}{n}} = 1 + d_n$. We have $r^n + 1 = (1+d_n)^n \geq 1 + nd_n$. Then $0 < d_n \leq \frac{r^n}{n}$ and as $\frac{r^n}{n} \to 0$ we have $d_n \to 0$ from which $(r^n+1)^{\frac{1}{n}} \to 1$.

12. Problem 16, page 67:
For (a), take $x_n = \frac{1}{n}$. For (b), take $x_n = n$.

13. Problem 18, page 67: These are all done as applications of **3.2.11**.
For (a), we have $\frac{(n+1)^2 a^{n+1}}{n^2 a^n} \to a$. As $a < 1$ we have $n^2 a^n \to 0$.
For (b), consider $\frac{n^2}{b^n} = n^2(\frac{1}{b})^n$. By (a), $n^2(\frac{1}{b})^n \to 0$ from which $(\frac{b^n}{n^2})$ is unbounded. Then $(\frac{b^n}{n^2})$ diverges.
For (c), as $\frac{b^{n+1}}{(n+1)!}\frac{n!}{b^n} = \frac{b}{n+1} \to 0$ we have $\frac{b^n}{n!} \to 0$.
For (d), we have $\frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!} = \left(\frac{n}{1+n}\right)^n = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e} < 1$. Then $\frac{n!}{n^n} \to 0$.

14. Problem 21, page 67:
If $x_n \to L$ we will prove $y_n \to L$. Let $\epsilon > 0$. There exists N such that if $n > N$, then $|x_n - L| < \frac{\epsilon}{2}$ and there exists M such that if $n > M$ then $|y_n - x_n| < \frac{\epsilon}{2}$. Choose $n > \max\{N, M\}$. Then $|y_n - L| = |(y_n - x_n) + (x_n - L)| \leq |y_n - x_n| + |x_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

15. Problem 4, page 74:
We have $x_{n+1}^2 = 2 + x_n$. If $x_n \to x$, then $x^2 = 2 + x$ from which $x > 0$ gives $x = 2$. We need to show that (x_n) converges. Note that if $x_k < 2$ then $x_{k+1}^2 < 4$ from which $x_{k+1} < 2$. By Mathematical Induction it follows that (x_n) is bounded above by 2. For $0 < y < 2$, we have $\sqrt{2+y} > y$. The sequence is increasing.

16. Problem 12, page 74:
$\frac{1}{1^2} + \frac{1}{2^2} + + \cdots + \frac{1}{n^2} + \leq 1 + \frac{1}{2\cdot 1} + \cdots + \frac{1}{n(n+1)} = 1 + \left(1 - \frac{1}{n}\right) < 2$. The sequence is bounded and increasing hence convergent.

17. Problem 13, page 74:
For (a), $(1+\frac{1}{n})^n(1+\frac{1}{n}) \to e \cdot 1 = e$.
For (b), $(1+\frac{1}{n})^{2n} = ((1+\frac{1}{n})^n)^2 \to e^2$.
For (c), $\left(1+\frac{1}{n+1}\right)^n = \frac{1}{1+\frac{1}{n+1}}\left(1+\frac{1}{n+1}\right)^{n+1} \to 1 \cdot e = e$.
For (d), $\left(\frac{n-1}{n}\right)^n = \frac{1}{\left(\frac{n}{n-1}\right)^n} = \frac{1}{\left(1+\frac{1}{n-1}\right)^n} = \frac{1}{\left(1+\frac{1}{n-1}\right)^{n-1}\left(1+\frac{1}{n-1}\right)} \to \frac{1}{e\cdot 1} = \frac{1}{e}$.