This note concerns the genesis of a solution to a simple introductory exercise for a course in Real Analysis.

Exercise: Prove that if $a > 0$, then

$$a + \frac{1}{a} \geq 2 \ .$$

The final version of the proof was put together in stages, in a fashion which differs little from conventional prose writing. There were issues of both form and content.

To begin with, I convinced myself of the truth of the statement to be proved. Looking at $f(x) = x + \frac{1}{x}$ for $x > 0$, first year calculus techniques give that f has a unique local minimum at $x = 1$. The function f is decreasing to the left of 1 and increasing to the right so that the local minimum at 1 is absolute. When $x = 1,\ f(x) = 2$, from which the result follows.

Since the techniques of differentiation, the Mean Value Theorem, etc., are not available for use, this proof though correct is not suitable in the context. Further, there is an aesthetic issue. The problem as posed appears very simple. That being the case, a simple, elementary solution is to be preferred.

How does one proceed? It is obvious that if $a \geq 2$, $a + \frac{1}{a} \geq 2$. That being the case, a proof for $0 < a < 2$ suffices. My first attempt considered the remaining cases.

- If $a = 1$, $1 + \frac{1}{1} = 2$.
- If $a < 1$, write $a = 1 - \epsilon$ where $0 < \epsilon < 1$. Then

$$\begin{aligned} & a + \frac{1}{a} \\ = \ & 1 - \epsilon + \frac{1}{1-\epsilon} \\ = \ & 1 - \epsilon + 1 + \epsilon + \epsilon^2 + \epsilon^3 + \cdots \\ = \ & 2 + \epsilon^2 + \epsilon^3 + \cdots \\ > \ & 2 \ . \end{aligned}$$

  As $|\epsilon| < 1$ the infinite geometric series, $1 + \epsilon + \epsilon^2 + \epsilon^3 + \cdots$, converges to $\frac{1}{1-\epsilon}$.

- If $1 < a < 2$ then $a = 1 + \epsilon$ where $0 < \epsilon < 1$. Then as above,

$$\begin{aligned} & a + \frac{1}{a} \\ = \; & 1 + \epsilon + \frac{1}{1+\epsilon} \\ = \; & 1 + \epsilon + 1 - \epsilon + \epsilon^2 - \epsilon^3 + \cdots \\ = \; & 2 + \epsilon^2 - \epsilon^3 + \cdots \\ = \; & 2 + \frac{\epsilon^2}{1+\epsilon} \\ > \; & 2 \; . \end{aligned}$$

What observations are possible from this? The similarity of the proofs for $a < 1$ and $a > 1$ suggests that one may be redundant. Given the pair of reciprocals a and $\frac{1}{a}$, one must be at least 1. Renaming if necessary, there is no reason not to assume $a \geq 1$ in the original problem.[1]

There is a more fundamental difficulty with the proof. Series, convergence and convergence of series have not yet been introduced, so that it is difficult to justify the proof steps which involve them. What is really required? The geometric series is a means to obtain the inequality

$$\frac{1}{1+\epsilon} \geq 1 - \epsilon \; .$$

The idea arises to prove this directly. This was simple enough as we can see below.

Solution: Since $a \cdot \frac{1}{a} = 1$, $a > 0$ from which $\frac{1}{a} > 0$, either a or $\frac{1}{a}$, its reciprocal, must be at least 1. Assume $a \geq 1$.
Note that for any ϵ, it follows that $\epsilon^2 \geq 0$. Then

$$(1 - \epsilon)(1 + \epsilon) = 1 - \epsilon^2 \leq 1 \; .$$

Provided $1 + \epsilon > 0$,

$$\frac{1}{1+\epsilon} \geq 1 - \epsilon \; .$$

[1]One could of course choose the alternate assumption, $a \leq 1$.

For $a \geq 1$, write $a = 1 + \epsilon$ where $\epsilon \geq 0$. Then $1 + \epsilon > 0$ and

$$\begin{aligned} & a + \frac{1}{a} \\ = \quad & 1 + \epsilon + \frac{1}{1+\epsilon} \\ \geq \quad & 1 + \epsilon + 1 - \epsilon \\ = \quad & 2 \ . \end{aligned}$$

This version of the proof is short, clear and to the point. What is hidden is the process that led to its formulation.

Postscript: A shift in perspective yields a different solution. Rewrite the inequality as

$$\frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \ .$$

The number 1 on the right is the geometric mean of a and $\frac{1}{a}$. The inequality follows as a special case of the general result that the geometric mean of positive numbers cannot exceed their arithmetic mean.

This is a proof suggested by Ganong. Focusing on the relation rather than the numbers a and $\frac{1}{a}$,

$$\begin{aligned} & a + \frac{1}{a} \geq 2 \\ \Leftrightarrow \quad & \frac{a^2+1}{a} \geq 2 \\ \Leftrightarrow \quad & a^2 + 1 \geq 2a \ \ (\text{since } a > 0) \\ \Leftrightarrow \quad & a^2 - 2a + 1 \geq 0 \\ \Leftrightarrow \quad & (a-1)^2 \geq 0, \end{aligned}$$

which of course holds.

Additional Problem:[2] Prove that if $a,\ b > 0$ and $a + b = 1$,

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2} \ .$$

[2] *Math. Trip. 1926*, given in G.H. Hardy, *A Course of Pure Mathematics*.

Note that the problem is symmetric in a and b. If the minimum value occurs only once, it must occur for $a = b = \frac{1}{2}$. Observe that

$$\left(\frac{1}{2}+2\right)^2+\left(\frac{1}{2}+2\right)^2=\frac{25}{2}\ .$$

For $0 \le \epsilon < \frac{1}{2}$,

$$\begin{aligned}
&\left(\tfrac{1}{2}-\epsilon+\tfrac{1}{\frac{1}{2}-\epsilon}\right)^2+\left(\tfrac{1}{2}+\epsilon+\tfrac{1}{\frac{1}{2}+\epsilon}\right)^2 \\
= \;& \left(\tfrac{1}{2}-\epsilon+2+4\epsilon+8\epsilon^2+16\epsilon^3+32\epsilon^4+64\epsilon^5+\cdots\right)^2 \\
& +\left(\tfrac{1}{2}+\epsilon+2-4\epsilon+8\epsilon^2-16\epsilon^3+32\epsilon^4-64\epsilon^5+\cdots\right)^2 \\
= \;& \left((\tfrac{5}{2}+8\epsilon^2+32\epsilon^4+\cdots)+(3\epsilon+16\epsilon^3+64\epsilon^5+\cdots)\right)^2 \\
& +\left((\tfrac{5}{2}+8\epsilon^2+32\epsilon^4+\cdots)-(3\epsilon+16\epsilon^3+64\epsilon^5+\cdots)\right)^2 \\
= \;& 2\left(\tfrac{5}{2}+8\epsilon^2+32\epsilon^4+\cdots\right)^2+2\left(3\epsilon+16\epsilon^3+64\epsilon^5+\cdots\right)^2\ ,
\end{aligned}$$

which is clearly minimal for $\epsilon = 0$. Rearrangement is permitted as the geometric series expansion converges absolutely.