York University

Faculty of Arts, Faculty of Science

Math 3110

Midterm Test 1

SOLUTIONS

Instructions:

1. There are 4 questions on 4 pages.
2. Answer all questions.
3. Your work must justify the answer you give.

Question	**Points**	**Marks**
1	6	
2	10	
3	6	
4	8	
Total	30	

1. (6 points) Let $f : A \to B$.

   (a) Prove that if $E \subseteq A$, then $E \subseteq f^{-1}(f(E))$.
   Answer:

$$\begin{aligned} & x \in E \\ \Rightarrow\ & f(x) \in f(E) \\ \Rightarrow\ & x \in f^{-1}(f(E)) \end{aligned}$$

   (b) Prove that if $F \subseteq B$, then $f(f^{-1}(F)) \subseteq F$.
   Answer:

$$\begin{aligned} & x \in f(f^{-1}(F)) \\ \Rightarrow\ & x = f(y) \ \text{ some } \ y \in f^{-1}(F) \\ \Rightarrow\ & x = f(y) \ \text{ where } \ f(y) \in F \\ \Rightarrow\ & x \in F \end{aligned}$$

2. (10 points)

(a) Prove that if $a \neq 0$ then $a^2 > 0$.
Answer:
If $a > 0$ then $a \in \mathbb{P}$ so that $a^2 = a \cdot a \in \mathbb{P}$.
If $a < 0$ then $-a \in \mathbb{P}$. Then $a^2 = (-a)^2 = (-a) \cdot (-a) \in \mathbb{P}$.

(b) Use Mathematical Induction to prove that all elements of $\mathbb{N}$ are positive.
Answer:
As $1 \neq 0$, by (a), $1^2 > 0$.
Assume for some $k \in \mathbb{N}$, $k > 0$, i.e., $k \in \mathbb{P}$. As $1 \in \mathbb{P}$, we have $k + 1 \in \mathbb{P}$.

3. (6 points) Define a set F_n by $T \in F_n$ provided $T \subseteq \mathbb{N}$, T is finite, and T has **at most** n elements. Prove that F_n is a countable set.

Answer:

It suffices to prove that F_n is the surjective image of a countable set.
Consider $S = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$, the n-fold Cartesian product of $\mathbb{N}$.
Map $S \to F_n$ by $(x_1, x_2, ..., x_n) \mapsto \{x_1, x_2, ..., x_n\}$.
This is a surjective map.
Since $S = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ is countable we are done.
To prove that S is countable let $S_m = \{(x_1, x_2, ..., x_n) \in S : x_1 + x_2 + \cdots + x_n = m\}$. The sets S_m are finite and we have $S = \bigcup_{m=0}^{\infty} S_m$.

4. (8 points) Prove that for $r \geq 0$,

$$r^2 \geq 4 \text{ if and only if } r \geq 2 \ .$$

Answer:
If $r \geq 2$, then $r > 0$ and $r^2 = r \cdot r \geq 2r$.
Similarly, as $r \geq 2$ and $2 > 0$, then $2r \geq 4$.
By transitivity, $r^2 \geq 4$.

For the converse, note that if $r^2 \geq 4$, then $r^2 - 4 = (r-2)(r+2) \geq 0$.
Then either $r - 2 \geq 0$ and $r + 2 \geq 0$ or $r - 2 \leq 0$ and $r + 2 \leq 0$.
In the latter case, we would have $r \leq 2$ which contradicts the assumption, $r \geq 0$.
In the former we must have $r \geq -2$ and $r \geq 2$.
As $r \geq 2 \Rightarrow r \geq -2$ these give $r \geq 2$ as required.

The end