York University

Faculty of Arts, Faculty of Science

Math 3110

Midterm Test 2

SOLUTIONS

Instructions:

1. There are 4 questions on 5 pages.
2. Answer all questions.
3. Your work must justify the answer you give.

Question	**Points**	**Marks**
1	4	
2	8	
3	8	
4	10	
Total	30	

1. (4 points) Prove that if $0 \leq x \leq 1$ and $0 \leq y \leq 1$, then $|x - y| \leq 1$.
   Answer: Here are two different solutions.

   From $0 \leq y \leq 1$ one obtains $-1 \leq -y \leq 0$.
   Adding this to $0 \leq x \leq 1$ gives $-1 \leq x - y \leq 1$ from which the result follows.

   This is a proof using the triangle inequality.
   We are given conditions equivalent to $|x - \frac{1}{2}| \leq \frac{1}{2}$ and $|y - \frac{1}{2}| \leq \frac{1}{2}$.
   Then $|x - y| = |(x - \frac{1}{2}) + (\frac{1}{2} - y| \leq |x - \frac{1}{2}| + |\frac{1}{2} - y| \leq \frac{1}{2} + \frac{1}{2} = 1$.

2. (8 points) Let $S \subseteq \mathbb{R}$ be nonempty. Prove that $u = \operatorname{lub} S$ if and only if for all $n \in \mathbb{N}$, we have $u + \frac{1}{n}$ is an upper bound for S, and $u - \frac{1}{n}$ is not an upper bound for S.
Answer:
If $u = \operatorname{lub} S$, we have $u \geq s$ for all $s \in S$. As $u + \frac{1}{n} > u$, we have $u + \frac{1}{n} \geq s$ for all $s \in S$, i.e., $u + \frac{1}{n}$ is an upper bound for S. Now $u - \frac{1}{n} < u$. Were $u - \frac{1}{n}$ an upper bound for S this would contradict the minimality of u.

Conversely, assume that for all n, $u + \frac{1}{n}$ is an upper bound for S, and $u - \frac{1}{n}$ is not an upper bound for S. We prove that u is an upper bound for S and that u is the least upper bound. If u were not an upper bound, there exists $x > u$, $x \in S$. By the Archimedean property, there exists n such that $\frac{1}{n} < x - u$. Then $x > u + \frac{1}{n}$ which contradicts the first part of the assumption. If u is not the least upper bound let $x < u$ be the least upper bound of S. By the Archimedean property, there exists n such that $\frac{1}{n} < u - x$. We have $x < u - \frac{1}{n}$. But as $u - \frac{1}{n}$ is not an upper bound for S, there exists $y \in S$, $x < u - \frac{1}{n} < y$. Then x is not an upper bound for S, a contradiction.

3. (8 points) Prove directly using the ϵ definition of limit that if (x_n) and (y_n) are sequences with $x_n \to 5$ and $y_n \to 7$ then the sequence $(2x_n + 3y_n)$ converges and $2x_n + 3y_n \to 31$.
Answer:
There exists M such that for $n > M$, we have $|x_n - 5| < \frac{\epsilon}{4}$.
There exists N such that for $n > N$, we have $|y_n - 7| < \frac{\epsilon}{6}$.

Take $n > \max(M, N)$. Then,

$$
\begin{aligned}
& |2x_n + 3y_n - 31| \\
= \; & |(2x_n - 10) + (3y_n - 21)| \\
\leq \; & |2x_n - 10| + |3y_n - 21| \\
= \; & 2|x_n - 5| + 3|y_n - 7| \\
\leq \; & 2 \cdot \frac{\epsilon}{4} + 3 \cdot \frac{\epsilon}{6} \\
= \; & \epsilon
\end{aligned}
$$

4. (10 points) Determine which of the following sequences converge and which do not. For those that converge find their limit. Justify your answer. For those that do not converge explain why not.

   (a) (n^2)
   Answer: The sequence diverges.
   We show that $\{n^2\}$ is unbounded. Note that for $n \geq 1$, $n^2 \geq n$. It suffices to show that $\{n\}$ is unbounded. If $\{n\}$ were bounded then there would exist $x \in \mathbb{R}$, $x \geq n$ for all n. This countradicts the Archimedian property.

   (b) (c^n) where $0 < c < 1$.
   Answer: The sequence converges to 0.
   Write $c = \frac{1}{1+d}$ for $d > 0$.
   Then $c^n = \frac{1}{(1+d)^n} < \frac{1}{1+nd}$ from which $0 < c^n < \frac{1}{1+nd}$.
   As $0 \to 0$ and $\frac{1}{1+nd} \to 0$, by the Squeeze Theorem, $c^n \to 0$.

(c) $(\sqrt{n+4}-\sqrt{n})$.
Answer: The sequence converges to 0.
Rationalize the numerator to obtain $(\sqrt{n+4}-\sqrt{n}) = \frac{4}{\sqrt{n+4}+\sqrt{n}} < \frac{4}{2\sqrt{n}} = \frac{2}{\sqrt{n}}$.
As $0 < \sqrt{n+4}-\sqrt{n} < \frac{4}{2\sqrt{n}}$ and $\frac{2}{\sqrt{n}} \to 0$ we have $\sqrt{n+4}-\sqrt{n} \to 0$.

(d) $\left((1+\frac{1}{n+2})^n\right)$.
Answer: The sequence converges to e.
$\left(1+\frac{1}{n+2}\right)^n = \frac{\left(1+\frac{1}{n+2}\right)^{n+2}}{\left(1+\frac{1}{n+2}\right)^2} \to \frac{e}{1^2} = e.$

The end