Math 3110 Homework 1 due September 20 at Noon SOLUTIONS

1. Let $f : A \to B$ and $T \subseteq B$.
   Prove that $f^{-1}(B \setminus T) = A \setminus f^{-1}(T)$.
   Answer:
   Note that $x \in f^{-1}(B \setminus T)$ if and only if for some $y \in B$, $y \notin T$ and $f(x) = y$.
   In addition, $x \in A \setminus f^{-1}(T)$ if and only if $x \in A$ and $f(x) \notin T$.

   If $x \in A \setminus f^{-1}(T)$ take $y = f(x)$. If $y \in T$ then $x \in f^{-1}(T)$ which is impossible. Then $y \notin T$ so that by the first observation, $x \in f^{-1}(B \setminus T)$. Then $A \setminus f^{-1}(T) \subseteq f^{-1}(B \setminus T)$.

   If $x \in f^{-1}(B \setminus T)$ then $f(x) \in B \setminus T$. Then $x \in A$ and $f(x) \notin T$ from which $x \in A \setminus f^{-1}(T)$. Then $f^{-1}(B \setminus T) \subseteq A \setminus f^{-1}(T)$ and equality follows.

2. Let $f : A \to B$ and $S \subseteq A$, $T \subseteq A$.

   (a) Prove that $f(S \cap T) \subseteq f(S) \cap f(T)$.
   Answer: Let $y \in f(S \cap T)$. Then there exists $x \in S \cap T$ such that $y = f(x)$. As $x \in S$, $y \in f(S)$. As $x \in T$, $y \in f(T)$ from which $y \in f(S) \cap f(T)$.

   (b) Prove that if f is injective, $f(S \cap T) = f(S) \cap f(T)$.
   Answer: As from (a) we have $f(S \cap T) \subseteq f(S) \cap f(T)$, we need only prove $f(S) \cap f(T) \subseteq f(S \cap T)$.

   Let $y \in f(S) \cap f(T)$. As $y \in f(S)$ there exists $x \in S$, $f(x) = y$. As $y \in f(T)$ there exists $x' \in T$, $f(x') = y$. As $f(x) = f(x')$ and f is injective, $x = x'$ and hence $x \in S \cap T$. Then as $y = f(x)$, $y \in f(S \cap T)$.

   (c) Give an example of f, S, T for which $f(S \cap T) \neq f(S) \cap f(T)$.
   Answer: Keeping (b) in mind, we need to choose f so that it is not injective.

   Take $A = \{0, 1, 2\}$ and $B = \{0, 1\}$.
   Define $f : A \to B$ by $f(0) = 0,\ f(1) = 1,\ f(2) = 0$.
   Take $S = \{0, 1\}$ and $T = \{1, 2\}$.
   Then $f(S \cap T) = \{1\}$ and $f(S) \cap f(T) = \{0, 1\}$.

3. For x a real number, let $\lfloor x \rfloor$ denote the largest integer less than or equal to x.

   (a) Prove that $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$.
   Hint: For n an integer, we have $\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$.
   Answer: Using the hint, from $\lfloor x \rfloor = n$ obtain $n \leq x < n + 1$, and from $\lfloor y \rfloor = n$ obtain $m \leq y < m + 1$. Adding gives $n + m \leq x + y < n + m + 2$.

   Either $n + m + 1 \leq x + y$ or $n + m + 1 > x + y$.

   In the first case, $n + m + 1 \leq x + y < n + m + 2$ from which

$\lfloor x+y \rfloor = n+m+1 \geq \lfloor x \rfloor + \lfloor y \rfloor$. In the second case $n+m \leq x+y < n+m+1$ from which $\lfloor x+y \rfloor = n+m \geq \lfloor x \rfloor + \lfloor y \rfloor$.

(b) Under what conditions does equality hold, i.e., is $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor x+y \rfloor$?
Answer: We have equality precisely when $m+n+1 > x+y$, i.e., $\lfloor x \rfloor + \lfloor y \rfloor + 1 > x+y$. Rewriting in terms of "fractional parts", gives the condition,
$(x - \lfloor x \rfloor) + (y - \lfloor y \rfloor) < 1$.

4. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x+1$. Are there any functions $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g = g \circ f$?
Answer: The condition to be satisfied is $g(x+1) = g(x)+1$. Any g with formula $g(x) = x+c$ for some $c \in \mathbb{R}$, commutes with f.

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a constant function. For which functions $g : \mathbb{R} \to \mathbb{R}$ does $f \circ g = g \circ f$?
Answer: Assume $f(x) = c$ for all x. The function g must satisfy $c = g(c)$.

(c) Suppose $f : \mathbb{R} \to \mathbb{R}$ with $f \circ g = g \circ f$ for *all* functions $g : \mathbb{R} \to \mathbb{R}$. Prove that for all x, $f(x) = x$.
Answer: Reverse the roles of f and g. If g is constant with value c, from $f \circ g = g \circ f$ we obtain $f(c) = c$. As c can be chosen arbitrarily, $f(c) = c$ for all $c \in \mathbb{R}$, i.e., f has formula, $f(x) = x$.