Math 3110 Homework 2 due Nov 4 and Nov 6 at Noon SOLUTIONS

1. Let S be a nonempty set of real numbers and let a be a nonzero real number. Suppose that $|x-a| < \dfrac{|a|}{2}$ for all $x \in S$. Prove that $|x| > \dfrac{|a|}{2}$ for all $x \in S$.
   Answer: From $|x-a| < \frac{|a|}{2}$ it follows that $a - \frac{|a|}{2} < x < a + \frac{|a|}{2}$.

   If $a \geq 0$ then $|a| = a$ from which $\frac{a}{2} < x < \frac{3a}{2}$ and $x > \frac{a}{2} = \frac{|a|}{2}$.

   If $a < 0$ then $|a| = -a$ from which $\frac{3a}{2} < x < \frac{a}{2}$. As $a < 0$ then $\frac{a}{2} < 0$ and $x < 0$. Finally $x < \frac{a}{2}$ gives $-x > \frac{-a}{2}$, i.e., $|x| > \frac{|a|}{2}$.

2. Let $x, a, y, b, \epsilon \in \mathbb{R}$, $\epsilon > 0$. Prove that if $|x-a| < \min\left(\dfrac{\epsilon}{2(|b|+1)}, 1\right)$ and $|y-b| < \dfrac{\epsilon}{2(|a|+1)}$, then $|xy - ab| < \epsilon$.
   Answer: By the triangle inequality,
   $|xy-ab| = |xy - bx + bx - ab| \leq |xy - bx| + |bx - ab| = |x||y-b| + |b||x-a|$.

   Observe that if $|x-a| < 1$, $|x| = |a + x - a| \leq |a| + |x-a| < |a| + 1$. Then,
$$\begin{aligned} |x||y-b| + |b||x-a| &< (|a|+1)\frac{\epsilon}{2(|a|+1)} + |b|\frac{\epsilon}{2(|b|+1)} \\ &= \frac{|a|+1}{|a|+1}\,\frac{\epsilon}{2} + \frac{|b|}{|b|+1}\,\frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ . \end{aligned}$$

3. Prove that if S is a nonempty bounded subset of $\mathbb{Z}$, both $\inf S$ and $\sup S$ belong to S.
   Answer: Assume that $\sup S = z$ with $z \notin S$. There exists $n \in S$ with $z - \frac{1}{2} < n < z$. Consider $m \in \mathbb{Z}$ with $m > n$. Then as $m \geq n+1$, we have $m > z + \frac{1}{2}$, i.e., $m > z$ from which $m \notin S$. Then n would be an upper bound for S, a contradiction as $\sup S > n$. The only possibility is $z \in S$.

   The proof that $\inf S$ belongs to S is identical. Assume $\inf S = u$ with $u \notin S$. There exists n with $u < n < u + \frac{1}{2}$. One obtains that n is a lower bound for S which contradicts $n > \inf S$.

4. Prove that for each $a \in \mathbb{R}$ and each $n \in \mathbb{N}$ there exists a rational number r_n with $|a - r_n| < \dfrac{1}{n}$. Do not just apply **2.4.8**, but rather prove this directly using the Archimedian Property for $\mathbb{R}$.
   Answer: We are looking for a rational number r_n with $a - \frac{1}{n} < r_n < a + \frac{1}{n}$.
   Let $s_n = n\,r_n$. We have $a - \frac{1}{n} < r_n < a + \frac{1}{n} \;\;\Leftrightarrow\;\; na - 1 < s_n < na + 1$.

   There exists an integer m such that $m \leq na - 1 < m + 1$ as follows. By the Archimedian property, $na-1$ cannot be an upper bound for $\mathbb{Z}$ from which $\{k \in \mathbb{Z} \,:\, k > na-1\}$ is nonempty. As any subset of $\mathbb{Z}$ which has a lower bound is well ordered (**this requires proof**), it has a least element $m+1$.

   Take $s_n = m+1$. Then $na - 1 < m + 1 = s_n$ and $s_n = m + 1 < m + 2 \leq na + 1$ from which $na - 1 < s_n < na + 1$ as required.