Math 3110 Homework 3 due December 3 at Noon

1. Prove that if a sequence (a_n) converges to $L > 0$, there exists $c > 0$ and $N \in \mathbb{N}$, such that, $a_n > c$ for all $n > N$.

   Answer: Take $c = \dfrac{L}{2}$.

   There exists N such that for $n > N$, $|a_n - L| < \dfrac{L}{2}$.

   Then
$$\frac{L}{2} = L - \frac{L}{2} < a_n < L + \frac{L}{2} = \frac{3L}{2} \ .$$

2. Let (a_n) be defined recursively by $a_1 = 2$ and $a_{n+1} = \dfrac{a_n}{2} + \dfrac{5}{a_n}$ for $n \geq 1$. Prove that (a_n) converges and find its limit.

   Answer: To show that (a_n) converges, we will show that for $n \geq 2$, (a_n) is nonincreasing. As (a_n) is bounded below by 0, it must converge.

   Observe that
$$a_{n+1} \leq a_n \ \Leftrightarrow\ \frac{a_n}{2} + \frac{5}{a_n} \leq a_n \ \Leftrightarrow\ \frac{5}{a_n} \leq \frac{a_n}{2} \ \Leftrightarrow\ a_n^2 \geq 10 \ .$$
   We use Mathematical Induction to prove that for $n \geq 2$, $a_N^2 \geq 10$.
   For $n = 2$, $a_n = \frac{7}{2}$ and direct calculation gives $a_n^2 \geq 10$.
   Assume that $a_n^2 \geq 10$. We have $a_{n+1}^2 \geq 10$ as follows.
$$a_{n+1}^2 = \left(\frac{a_n}{2} + \frac{5}{a_n}\right)^2 = \frac{a_n^2}{4} + 5 + \frac{25}{a_n^2}$$
   from which
$$a_{n+1}^2 \geq 10 \ \Leftrightarrow\ \left(\frac{a_n}{2} + \frac{5}{a_n}\right)^2 - 10 \geq 0 \ \Leftrightarrow\ \left(\frac{a_n}{2} - \frac{5}{a_n}\right)^2 \geq 0 \ .$$
   As $a_n^2 \geq 10$ for $n \geq 2$, the sequence (a_n) is nonincreasing for $n \geq 2$.

   The limit L satisfies $L = \dfrac{L}{2} + \dfrac{5}{L}$. Since $L^2 = 10$ and since $L \geq 0$, we have $L = \sqrt{10}$.

3. Prove that if $a_n \to L$ then $\dfrac{a_1 + a_2 + \ldots + a_n}{n} \to L$.

   Answer: Let $\epsilon > 0$.
   There exists N_1 such that for $n > N_1$, $|a_n - L| < \frac{\epsilon}{2}$.
   Let M be such that for $n \leq N_1$, $|a_n - L| < M$.
   Let $n > N_1$. Then
$$\begin{aligned}
&\left|\frac{a_1 + a_2 + \ldots + a_n}{n} - L\right| \\
=\ &\left|\frac{a_1 + a_2 + \ldots + a_n}{n} - \frac{nL}{n}\right| \\
\leq\ &\left(\frac{|a_1 - L|}{n} + \cdots + \frac{|a_{N_1} - L|}{n}\right) + \left(\frac{|a_{N_1+1} - L|}{n} + \cdots + \frac{|a_n - L|}{n}\right) \\
<\ &\frac{N_1 M}{n} + \frac{n - N_1}{n}\frac{\epsilon}{2} \\
<\ &\frac{N_1 M}{n} + \frac{\epsilon}{2} \ .
\end{aligned}$$

By the Archimedean Property, there exists N_2 such that for $n > N_2$, $\dfrac{N_1 M}{n} < \dfrac{\epsilon}{2}$.
Choose $n \geq \max\{N_1, N_2\}$. Then

$$\left|\frac{a_1 + a_2 + \ldots + a_n}{n} - L\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ .$$

4. Prove the following statement about Cauchy sequences without using the fact that a Cauchy sequence of real numbers converges.

   If a subsequence of a Cauchy sequence (a_n) converges, then the sequence (a_n) converges.
   Answer: Let $\epsilon > 0$.
   Since (a_n) is Cauchy, there exists M such that for $n, m > M$, we have $|x_m - x_n| < \frac{\epsilon}{2}$. If (a_{n_k}) is a subsequence which converges to L, there exists N such that if $k > N$ we have $|a_{n_k} - L| < \frac{\epsilon}{2}$.
   Let $n > \max\{M, N\}$. Fix $k > \max\{M, N\}$. Then $n_k > k > \max\{M, N\}$ and

$$|a_n - L| = |x_n - a_{n_k} + a_{n_k} - L| \leq |x_n - a_{n_k}| + |a_{n_k} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ .$$