Math 3110 A
Quiz 1 October 4, 2002
SOLUTIONS

1. Let $f : X \to Y$.

   (a) (4 points) Prove that if $A \subseteq X$, then $f(X) \setminus f(A) \subseteq f(X \setminus A)$.

   Answer: If $y \in f(X) \setminus F(A)$ then there exists $x \in X$ with $f(x) = y$.
   If $x \in A$, then $y = f(x) \in f(A)$.
   But $y \notin f(A)$, so $x \notin A$, giving $x \in X \setminus A$ and $y = f(x) \in f(X \setminus A)$.

(b) (1 point) What is meant by the statement, "f is injective".

Answer: Given $x, x' \in X$, if $f(x) = f(x')$ then $x = x'$.

Alternatively, given $x, x' \in X$, if $x \neq x'$ then $f(x) \neq f(x')$.

(c) (4 points) Prove that if f is injective and $A \subseteq X$, then $f(X \setminus A) \subseteq f(X) \setminus f(A)$.

Answer: Let $y \in f(X \setminus A)$. There exists $x \in X,\ x \notin A$ such that $y = f(x)$. Then $y \in f(X)$.

We need to show $y \notin f(A)$. Assume otherwise, i.e., that there exists $x' \in A$ such that $f(x') = y$. Then $f(x') = f(x)$ which by injectivity gives $x' = x$. But $x \notin A$ which is a contradiction.

Then $y \in f(X),\ y \notin f(A)$, i.e., $y \in f(X) \setminus f(A)$.

(d) (3 points) Prove that if $f(X \setminus A) = f(X) \setminus f(A)$ for **all** $A \subseteq X$, then f is injective. **Suggestion:** Assume that there exist x, x' with $f(x) = f(x')$ and $x \neq x'$. Choose a suitable A and obtain a contradiction.

Answer: Assume x, x' exist with $f(x) = f(x')$ and $x \neq x'$. Let $A = \{x\}$.

As $x' \in X \setminus A$, and $f(X \setminus A) = f(X) \setminus f(A)$, we have $f(x') \in f(X) \setminus f(A)$. In particular, $f(x') \notin f(A)$. But $f(x) \in f(A)$ and $f(x) = f(x')$, a contradiction.

2. Let $x_1 = \frac{3}{4}$. For $n > 1$ define $x_{n+1} = 1 - \sqrt{1 - x_n}$.

   (a) (6 points) Use Mathematical Induction to prove that this definition makes sense, i.e., prove that for all $n \in \mathbb{N}$, $0 < x_n < 1$.
   Note: You may use without proof that $0 < a < 1$ if and only if $0 < \sqrt{a} < 1$.

   Answer: For $n = 1$ just observe that $0 < \frac{3}{4} < 1$.

   Assume that for some k, $0 < x_k < 1$. Consider $x_{k+1} = 1 - \sqrt{1 - x_k}$.
   As $0 < x_k < 1$, then $0 < 1 - x_k < 1$ from which $0 < \sqrt{1 - x_k} < 1$.
   Then $0 < 1 - \sqrt{1 - x_k} < 1$, i.e., $0 < x_{k+1} < 1$ as required.

(b) (2 points) Prove that for all n, $x_{n+1} < x_n$.
Standard properties of inequalities may be used without proof.

Answer: Keeping in mind that $1 - x_n > 0$, observe that

$$\begin{aligned} & x_{n+1} < x_n \\ \Leftrightarrow\ & 1 - \sqrt{1 - x_n} < x_n \\ \Leftrightarrow\ & 1 - x_n < \sqrt{1 - x_n} \\ \Leftrightarrow\ & 1 - 2x_n + x_n^2 < 1 - x_n \\ \Leftrightarrow\ & x_n^2 < x_n \ . \end{aligned}$$

But given that $0 < x_n < 1$, multiplication by x_n gives $x_n^2 < x_n$ from which $x_{n+1} < x_n$ as required.