Math 3110 A
Quiz 2 November 8, 2002
SOLUTIONS

1. Suppose that $x, a, y, b \in \mathbb{R}$ are such that for some $\epsilon > 0$, $|x - a| < \epsilon$ and $|y - b| < \epsilon$.

   (a) (2 points) Prove that,
   $$|x| < |a| + \epsilon \ .$$

   Answer: By the triangle inequality,
   $$|x| = |a + x - a| \leq |a| + |x - a| < |a| + \epsilon \ .$$

   (b) (4 points) Prove that,
   $$|xy - ab| < (|a| + |b|)\epsilon + \epsilon^2 \ .$$

   Hint: $xy - ab = xy - xb + xb - ab$.
   Answer: By the triangle inequality,
   $$\begin{aligned} |xy - ab| &= |xy - xb + xb - ab| \\ &\leq |xy - xb| + |xb - ab| \\ &= |x||y - b| + |b||x - a| \\ &< (|a| + \epsilon)\epsilon + |b|\epsilon \\ &= (|a| + |b|)\epsilon + \epsilon^2 \end{aligned}$$

2. Let $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$.

(a) (1 point) What is meant by the statement, "1 is an upper bound for S".
Answer:
For all $x \in S$, we have $1 \geq S$.

(b) (2 points) Using the Order Properties of $\mathbb{R}$, prove that 1 is an upper bound for S.
Answer:
As $n > 0$, $\frac{1}{n} > 0$ and $1 - \frac{1}{n} < 1$.

(c) (1 point) State the Archimedian Property for $\mathbb{R}$.
Answer:
Given $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $n > x$.

(d) (1 point) What is meant by the statement, "1 is the least upper bound of S".
Answer:
1 is an upper bound for S and given $\epsilon > 0$ there exists $s \in S$ such that $1 - \epsilon < s$, i.e., anything smaller than 1 is not an upper bound for S.

(e) (4 points) Use the Archimedian Property for $\mathbb{R}$ to prove that 1 is the least upper bound of S.
Answer:
Let $\epsilon > 0$. By the Archimedian Property there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\epsilon}$ from which $\epsilon > \frac{1}{n}$. Then $1 - \epsilon < 1 - \frac{1}{n}$ and as $1 - \frac{1}{n} \in S$ we are done.