York University

Faculty of Arts, Faculty of Science

Math 3110

Midterm Test 1

SOLUTIONS

Instructions:

1. There are 4 questions on 4 pages.
2. Answer all questions.
3. Your work must justify the answer you give.

Question	**Points**	**Marks**
1	10	
2	9	
3	5	
4	6	
Total	30	

1. Let $f : A \to B$ and $g : B \to A$ such that $g \circ f$ is bijective.

   (a) i. (1 point) What is meant by the statement, "f is injective"?

   Answer:
   Whenever $f(a) = f(a')$ it follows that $a = a'$.

   ii. (4 points) Prove that f is injective.

   Answer:
   If $f(a) = f(a')$ then $g(f(a)) = g(f(a'))$ from which $(g \circ f)(a) = (g \circ f)(a')$. But as $g \circ f$ is bijective, it is injective, from which $a = a'$.

   i. (1 point) What is meant by the statement, "g is surjective"?

   Answer:
   Given any $a \in A$ there exists $b \in B$ such that $g(b) = a$.

   ii. (4 points) Prove that g is surjective.

   Answer:
   Since $g \circ f$ is bijective, we know that $g \circ f$ is surjective.
   Let $a \in A$. There exists a' such that $(g \circ f)(a') = g(f(a')) = a$. Set $b = f(a')$. Then $g(b) = a$.

2. Let $x_1 = 2$ and $x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbb{N}$.

(a) (5 points) Use Mathematical Induction to prove that for all $n \in \mathbb{N}$, $x_n > 1$.
Answer:
For $n = 1$ observe that $2 > 1$.
Assume $x_k > 1$.
Then $\frac{1}{x_k} < 1$, from which,

$$x_{k+1} = 2 - \frac{1}{x_k} = 1 + (1 - \frac{1}{x_k}) > 1 \ .$$

(b) (4 points) Prove that for all $n \in \mathbb{N}$, $\quad x_{n+1} < x_n$.
Answer:
Keeping in mind that $x_n > 0$,

$$\begin{aligned} & x_{n+1} < x_n \\ \Leftrightarrow \quad & 2 - \frac{1}{x_n} < x_n \\ \Leftrightarrow \quad & 2x_n - 1 < x_n^2 \\ \Leftrightarrow \quad & x_n^2 - 2x_n + 1 > 0 \\ \Leftrightarrow \quad & (x_n - 1)^2 > 0 \\ \Leftrightarrow \quad & x_n \neq 1 \end{aligned}$$

which holds as $x_n > 1$ for all n.

3. (a) (1 point) Explain what is meant by the statement, "the set S is denumerable". Your answer should be expressed in terms of the existence of a function with particular properties.

Answer:

S is denumerable means there exists a bijection $f : \mathbb{N} \to S$.

(b) (4 points) Prove that the set of odd natural numbers, $\{1, 3, 5, 7, \ldots\}$ is a denumerable set by defining a suitable function and **proving** it has the required properties.

Answer:

Define $f : \mathbb{N} \to \{1, 3, 5, 7, \ldots\}$ by $f(n) = 2n - 1$.

f is clearly surjective.

To show that f is injecitve observe that if $2n - 1 = 2n' - 1$, then $2n = 2n'$, from which $n = n'$.

4. (a) (2 points) If $a \in \mathbb{R}$, what property characterizes the expression $-a$?
Answer:
$-a \in \mathbb{R}$ with $a + (-a) = 0$.

(b) (2 points) Prove that $-a$ is uniquely determined by a.
Answer:
Assume that $a + b = 0$ and $a + c = 0$. We need to prove that $b = c$.
We have,

$$\begin{aligned}
(c+a)+b &= c+(a+b), \\
(a+c)+b &= c+0, \\
0+b &= c, \\
b+0 &= c, \\
b &= c\ .
\end{aligned}$$

(c) (2 points) Prove that if $a, b \in \mathbb{R}$, then $-(a+b) = (-a) + (-b)$.
Answer:

We have,

$$\begin{aligned}
& (a+b)+(-a)+(-b) \\
= \;& a+(-a)+b+(-b) \\
= \;& 0+0 \\
= \;& 0\ .
\end{aligned}$$

The end