York University

Faculty of Arts, Faculty of Science

Math 3110

Midterm Test 2

SOLUTIONS

Instructions:

1. There are 3 questions on 4 pages.
2. Answer all questions.
3. Your work must justify the answer you give.

Question	**Points**	**Marks**
1	14	
2	6	
3	10	
Total	30	

1. (a) (6 points) Prove that if $|x-1|<1$, then $|x^3+x-2|<8|x-1|$.
Hint: $x^3+x-2=(x^2+x+2)(x-1)$.

Answer: If $|x-1|<1$, then $-1<x-1<1$ from which $0<x<2$ and $|x|<2$.

$$
\begin{aligned}
|x^3+x-2| &= |(x^2+x+2)(x-1)| \\
&= |x^2+x+2||x-1| \\
&\le (|x|^2+|\,|x|+|2|)|x-1| \\
&< (4+2+2)|x-1| \\
&= 8|x-1|\ .
\end{aligned}
$$

(b) (2 points) Given that (x_n) is a sequence, explain what is meant by "y_n converges to 2."

Answer: Given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|x_n - 1| < \epsilon$.

(c) (6 points) Let (x_n) be a sequence for which x_n converges to 1. Define a sequence (y_n) by $y_n = x_n^2 + x_n$.
Prove that (y_n) converges to 2.

Answer: Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that if $n > N$ then $|x_n - 1| < \min\left\{1, \frac{\epsilon}{8}\right\}$.
Then if $n > N$, we have $|x_n - 1| < 1$ and

$$|y_n - 2| = |x_n^2 + x_n - 2| < 8|x_n - 1| = 8 \cdot \frac{\epsilon}{8} = \epsilon \ .$$

2. (6 points) Prove that if $I = (-\infty, b)$ then $\sup I = b$.
Hint: Prove that b is an upper bound and that if $y < b$ then y is not an upper bound for I.

Answer:
If $x \in (-\infty, b)$ then in particular $x < b$ from which b is an upper bound for I.
Assume that $y < b$ is an upper bound for I.
As $y < b$, we have $y + y < y + b < b + b$ and $y = \dfrac{y+y}{2} < \dfrac{y+b}{2} < \dfrac{b+b}{2} = b$.
As $\dfrac{y+b}{2} < b$ we have $\dfrac{y+b}{2} \in I$.
Since $y < \dfrac{y+b}{2}$, y is not an upper bound for I.
The only possibility is $b = \sup I$.

3. (a) (2 points) Carefully state the Archimedian Property for $\mathbb{R}$.
Answer: Given $z \in \mathbb{R}$ there exists $n \in \mathbb{N}$ with $n > z$.

(b) (3 points) Prove that if $I_n = [n, \infty)$ then $\bigcap_{n \in \mathbb{N}} I_n = \emptyset$.

Answer: If $z \in \bigcap_{n \in \mathbb{N}} I_n$ then for all $n \in \mathbb{N}$ we must have $z \geq n$. This contradictions the Archimedian Property for $\mathbb{R}$.

(c) (5 points) Prove that if (x_n) is a sequence such that $|x_n| \leq \frac{1}{n}$ for all n, then (x_n) converges to 0.
Answer:
Let $\epsilon > 0$. By the Archimedean Property for $\mathbb{R}$, there exists N such that $N > \dfrac{1}{\epsilon}$ from which $\dfrac{1}{N} < \epsilon$.

Let $n > N$. Then,

$$|x_n - 0| = |x_n| \leq \frac{1}{n} < \frac{1}{N} < \epsilon \ .$$

The end