SOLUTIONS TO FINAL EXAM

Show all work necessary to justify each of your answers.

1. [2 Marks] Find the domain of the function

$$f(x) = \sqrt{\frac{x}{x^2-1}} .$$

Express your answer using interval notation.

Solution The function $f(x)$ is defined when:

(1) the denominator $(x-1)(x+1)$ of this rational function is nonzero.

(2) the function $\frac{x}{(x-1)(x+1)} \geq 0$ so that the square root is defined;

Condition (1) is met for $x \neq \pm 1$ while condition (2) is met for $x \in (-1, 0] \cup (1, \infty)$. Hence the domain of f is $(-1, 0] \cup (1, \infty)$.

2. [3 Marks] The one–to–one functions f and g satisfy:

$$\begin{aligned} f(1) &= 3 \qquad & g(1) &= 2 \\ f(2) &= 1 & g(2) &= 3 \\ f(3) &= 2 & g(3) &= 1 \end{aligned}$$

(a) Find $(f \circ g)(2)$.
(b) Find $(g \circ f^{-1})(2)$.
(c) Find $(f^{-1} \circ f)(4)$.

Solution **(a)** $(f \circ g)(2) = f(g(2)) = f(3) = 2$.
(b) $(g \circ f^{-1})(2) = g(f^{-1}(2)) = g(3) = 1$.
(c) $(f^{-1} \circ f)(4) = 4$.

3. [4 Marks] **(a)** Find $\arcsin\left(\sin \frac{7\pi}{12}\right)$.
(b) Find $\sin\left(\arcsin \frac{7}{12}\right)$.

Solution **(a)** $\arcsin\left(\sin \frac{7\pi}{12}\right) = \arcsin\left[\sin\left(\pi - \frac{7\pi}{12}\right)\right] = \arcsin\left[\sin\left(\frac{5\pi}{12}\right)\right] = \frac{5\pi}{12}$.
(b) $\sin\left(\arcsin \frac{7}{12}\right) = \frac{7}{12}$.

4. [3 Marks] Evaluate the following limit.

$$\lim_{x \to 1} \frac{\sqrt[4]{x} - 1}{x - 1}$$

Do not use L'Hopital's Rule.

Solution Factor the denominator:

$$\begin{aligned} \lim_{x \to 1} \frac{\sqrt[4]{x} - 1}{x - 1} &= \lim_{x \to 1} \frac{\sqrt[4]{x} - 1}{(\sqrt{x} - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{\sqrt[4]{x} - 1}{(\sqrt[4]{x} - 1)(\sqrt[4]{x} + 1)(\sqrt{x} + 1)} \\ &= \lim_{x \to 1} \frac{1}{(\sqrt[4]{x} + 1)(\sqrt{x} + 1)} = \frac{1}{(1+1)(1+1)} = \frac{1}{4} . \end{aligned}$$

5. [9 Marks] Evaluate the following limits using any method.

(**a**) $\lim_{\theta \to \infty} \frac{\sin \theta}{\theta}$ (**b**) $\lim_{x \to 0+} \frac{x}{\sin^2 \sqrt{x}}$ (**c**) $\lim_{x \to \infty} \frac{\arctan x}{\sqrt{x}}$

Solution (**a**) Since $-1 \le \sin\theta \le 1$ for every angle θ,

$$\frac{-1}{\theta} \le \frac{\sin\theta}{\theta} \le \frac{1}{\theta}.$$

Since

$$\lim_{\theta\to\infty} \frac{-1}{\theta} = \lim_{\theta\to\infty} \frac{1}{\theta} = 0,$$

it follows from the Pinching Theorem that $\lim_{\theta\to\infty} \frac{\sin\theta}{\theta} = 0$ too.

(**b**) Let $\theta = \sqrt{x}$. Then

$$\lim_{x\to 0+} \frac{x}{\sin^2\sqrt{x}} = \lim_{\theta\to 0+} \frac{\theta^2}{\sin^2\theta} = \left(\lim_{\theta\to 0+} \frac{\theta}{\sin\theta}\right)^2 = (1)^2 = 1 .$$

Alternatively, this limit is the indeterminate form $\frac{0}{0}$. Hence L'Hopital's Rule applies.

$$\lim_{x\to 0+} \frac{x}{\sin^2\sqrt{x}} = \lim_{x\to 0+} \frac{D(x)}{D(\sin^2\sqrt{x})} = \lim_{x\to 0+} \frac{1}{2\sin\sqrt{x}\cos\sqrt{x}\frac{1}{2\sqrt{x}}} = \lim_{x\to 0+} \frac{\sqrt{x}}{\sin\sqrt{x}\cos\sqrt{x}} .$$

The preceding limit is also the indeterminate form $\frac{0}{0}$. Thus we apply L'Hopital's Rule again.

$$\begin{aligned}\lim_{x\to 0+} \frac{x}{\sin^2\sqrt{x}} &= \lim_{x\to 0+} \frac{D(\sqrt{x})}{D(\sin\sqrt{x})} \lim_{x\to 0+} \frac{1}{\cos\sqrt{x}} = \lim_{x\to 0+} \frac{D(\sqrt{x})}{\cos\sqrt{x}D(\sqrt{x})}(1) \\ &= \lim_{x\to 0+} \frac{1}{\cos\sqrt{x}} = \frac{1}{1} = 1 .\end{aligned}$$

(**c**) This limit is not an indeterminate form. In fact,

$$\lim_{x\to\infty} \frac{\arctan x}{\sqrt{x}} = \frac{\pi/2}{\infty} = 0 .$$

6. [2 Marks] State the Intermediate Value Theorem.

Solution Let f be a continuous function with domain an interval I. If $A < B$ are two numbers in the range of f, then the entire closed interval $[A, B]$ is a subset of the range of f.

7. [2 Marks] Consider the function f with domain $(-\pi, 2\pi)$ defined by:

$$f(x) = \begin{cases} \frac{|x \sin x|}{\sin x} & \textit{if } -\pi < x < 2\pi,\ x \neq 0,\ x \neq \pi \\ \pi & \textit{if } x = 0 \textit{ or } x = \pi \end{cases}$$

Find all points where f has a removable discontinuity, and find all points where f has an essential discontinuity.

Solution Observe that when $\sin x \geq 0$, i.e. for $x \in [0, \pi]$, we have $|\sin x| = \sin x$ and $f(x) = x$. On the other hand, when $\sin x < 0$, i.e. for $x \in (-\pi, 0) \cup (\pi, 2\pi)$, we have $|\sin x| = -\sin x$ and $f(x) = -|x|$.
Hence f is continuous for x in its domain except possibly at $x = 0$, and at $x = \pi$. When $x = 0$:

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} -|x| = \lim_{x \to 0^-} -x = 0 \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} x = 0 \end{aligned}$$

Since the left and right limits have the same value zero, the two–sided limit exists and equals zero. However, $f(0) = \pi$. Hence f has a removable discontiuity at $x = 0$. When $x = \pi$:

$$\begin{aligned} \lim_{x \to \pi^-} f(x) &= \lim_{x \to \pi^-} x = \pi \\ \lim_{x \to \pi^+} f(x) &= \lim_{x \to \pi^+} -|x| = \lim_{x \to \pi^+} -x = -\pi \, . \end{aligned}$$

Since the left and right limits have different values, the two–sided limit does not exist and f has an essential discontinuity at $x = \pi$.

8. [2 Marks] Let $f(x) = |x^4 - 2x^2 + 1|$. Find all values of x for which $f'(x)$ exists.

Solution Observe that $f(x) = |(x^2 - 1)^2| = (x^2 - 1)^2 = x^4 - 2x^2 + 1$ is a polynomial. Hence $f'(x)$ exists for every $x \in \Re$.

9. [2 Marks] Consider the function

$$f(x) = (\arctan x)^{2/3} \, .$$

Does f have a horizontal tangent, vertical tangent, ordinary cusp or vertical cusp at $x = 0$? Justify your answer.

Solution By the chain rule:

$$f'(x) = \frac{2}{3}(\arctan x)^{-1/3} \frac{1}{1 + x^2} = \frac{2}{3(1 + x^2)(\arctan x)^{1/3}}$$

Hence $f'(x)$ does not exist at $x = 0$. However,

$$\begin{aligned} \lim_{x \to 0^-} f'(x) &= \lim_{x \to 0^-} \frac{2}{3(1 + x^2)} \lim_{x \to 0^-} \frac{1}{(\arctan x)^{1/3}} = \frac{2}{3}(-\infty) = -\infty \\ \lim_{x \to 0^+} f'(x) &= \lim_{x \to 0^+} \frac{2}{3(1 + x^2)} \lim_{x \to 0^+} \frac{1}{(\arctan x)^{1/3}} = \frac{2}{3}(+\infty) = +\infty \end{aligned}$$

Thus f has a vertical cusp at $x = 0$.

10. [3 Marks] Find $f'(x)$ where

$$f(x) = \frac{\arcsin x}{\sqrt{1 - x^2}} \, .$$

Simplify your answer.

Solution By the quotient rule:

$$\begin{aligned} f'(x) &= \frac{D(\arcsin x)\sqrt{1-x^2} - (\arcsin x)D(\sqrt{1-x^2})}{(\sqrt{1-x^2})^2} \\ &= \frac{\left(\frac{1}{\sqrt{1-x^2}}\right)\sqrt{1-x^2} - (\arcsin x)\left(\frac{-2x}{2\sqrt{1-x^2}}\right)}{1-x^2} \cdot \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} = \frac{\sqrt{1-x^2} + x\arcsin x}{(1-x^2)^{3/2}} \end{aligned}$$

11. [3 Marks] Find $\frac{dy}{dx}$ where

$$\frac{1}{1+\sqrt{y}} + \sqrt{xy} = 1 .$$

Solution Take the derivative of the given equation:

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{1+\sqrt{y}} + \sqrt{xy}\right) &= \frac{d}{dx}(1) \\ -\frac{1}{(1+\sqrt{y})^2}\frac{d}{dx}(1+\sqrt{y}) + \frac{1}{2\sqrt{xy}}\frac{d}{dx}(xy) &= 0 \\ -\frac{1}{(1+\sqrt{y})^2}\left(0 + \frac{1}{2\sqrt{y}}\frac{dy}{dx}\right) + \frac{1}{2\sqrt{xy}}\left((1)y + x\frac{dy}{dx}\right) &= 0 \end{aligned}$$

Now solve this linear equation for $\frac{dy}{dx}$.

$$\frac{dy}{dx}\left(-\frac{1}{(1+\sqrt{y})^2}\frac{1}{2\sqrt{y}} + \frac{x}{2\sqrt{xy}}\right) = -\frac{y}{2\sqrt{xy}}$$

Simplify this equation by multiplying it by $2\sqrt{y}$.

$$\begin{aligned} \frac{dy}{dx}\left(-\frac{1}{(1+\sqrt{y})^2} + \sqrt{x}\right) &= -\frac{y}{\sqrt{x}} \\ \frac{dy}{dx}\left(\frac{-1+\sqrt{x}(1+\sqrt{y})^2}{(1+\sqrt{y})^2}\right) &= -\frac{y}{\sqrt{x}} \\ \frac{dy}{dx} &= \frac{y(1+\sqrt{y})^2}{\sqrt{x}\left[1-\sqrt{x}(1+\sqrt{y})^2\right]} \end{aligned}$$

12. [2 Marks] Let $f(x) = \arctan(x^3)$.
(a) Why is f one–to–one?
(b) Find $Df^{-1}\left(\frac{\pi}{4}\right)$.

Solution **(a)** By the chain rule:

$$f'(x) = \frac{1}{1+(x^3)^2}(3x^2) = \frac{3x^2}{1+x^6} \geq 0$$

for all x. Therefore f is an increasing function for all x. In particular if $a < b$, then $f(a) < f(b)$. Thus f is one–to–one.

(b) Note that $f(1) = \arctan 1 = \frac{\pi}{4}$. Thus

$$Df^{-1}\left(\frac{\pi}{4}\right) = \frac{1}{Df(f^{-1}(\pi/4))} = \frac{1}{Df(1)} = \frac{1}{3/2} = \frac{2}{3}.$$

13. [2 Marks] f and g are differentiable functions with domain $\Re$ which satisfy:

$f(1) = 3$	$f'(1) = 5$	$g(1) = 2$	$g'(1) = -4$
$f(2) = 1$	$f'(2) = -6$	$g(2) = 3$	$g'(2) = 8$
$f(3) = 2$	$f'(3) = 7$	$g(3) = 1$	$g'(3) = -9$

Find $D(g \circ f)(2)$.

Solution $D(g \circ f)(2) = Dg(f(2)) \cdot Df(2) = Dg(1) \cdot Df(2) = (-4)(-6) = 24$.

14. [2 Marks] State the Mean Value Theorem.

Solution Let f be a continuous function with domain $[a, b]$. Assume that $f'(x)$ exists for all $x \in (a, b)$. Then there is at least one number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

15. [3 Marks] Find and identify the critical points of the function

$$f(x) = (x^2 - x - 2)^{2/3}.$$

Solution By the chain rule:

$$f'(x) = \frac{2}{3}(x^2 - x - 2)^{-1/3}(2x - 1) = \frac{2(2x - 1)}{(x^2 - x - 2)^{1/3}} = \frac{2(2x - 1)}{(x - 2)^{1/3}(x + 1)^{1/3}}.$$

Hence f has a critical point at $x = \frac{1}{2}$ where $f'(x) = 0$ and at $x = 2$, $x = -1$ where $f'(x)$ does not exist. The sign of $f'(x)$ is depicted on the diagram below. By the first Derivative Test, f has a local minimum at $x = -1$, a loal maximum at $x = \frac{1}{2}$ and a local minimum at $x = 2$.

- - - - - DNE + + + + 0 - - - - - - - DNE + + + sign f'(x)

$\longrightarrow x$

decreasing -1 increasing $\frac{1}{2}$ decreasing 2 increasing

local min local max local min

16. [2 Marks] Find all the inflection points of the function $f(x) = (x^2 - 28)^4$.

Solution By the chain rule

$$\begin{aligned} f'(x) &= 4(x^2 - 28)^3(2x) = 8x(x^2 - 28)^3 \\ f''(x) &= 8(x^2 - 28)^3 + 8x3(x^2 - 28)^2(2x) = 8(x^2 - 28)^2(x^2 - 28 + 6x^2) \\ &= 8(x^2 - 28)^2(7x^2 - 28) = 56(x^2 - 28)^2(x^2 - 4) \\ &= 56(x^2 - 28)^2(x - 2)(x + 2) \end{aligned}$$

The sign of $f''(x)$ is sketched on the number line below. Since the concavity of f changes from up to down at $x = -2$ we have an inflection point there.
Since the concavity of f changes from down to up at $x = 2$ we have an inflection point there.
(Note that there is neither an inflection point at $x = -\sqrt{28}$ nor at $x = \sqrt{28}$ because f is concave up at $x = -\sqrt{28}$ and at $x = \sqrt{28}$, i.e the concavity does not change at these points.)

+ + + + *0* + + + + *0* - - - - *0* + + + + *0* + + + + *sign f"(x)*

x

concave up $-\sqrt{28}$ *concave up* *-2* *concave down* *2* *concave up* $\sqrt{28}$ *concave up*

infl. pt. *infl. pt.*

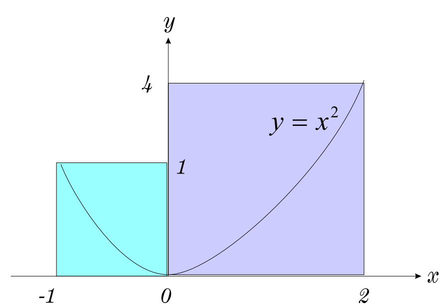
17. [2 Marks] Find the minimum and maximum values, if they exist, of the function $f(x) = 1-x^2$ with domain $(-1, 2]$.

Solution Since $f'(x) = -2x$, the function f has one critical point at $x = 0$ with $f(0) = 1$. By the First Derivative Test f has a local maximum at $x = 0$. The limit of $f(x)$ as x approaches the left endpoint -1 of its doman is zero, and the value of f at the right endpoint 2 of its domain is -3. Thus f has a maximum value of 1 at $x = 0$ and a minimum value of -3 at $x = 2$.

18. [2 Marks] Find the Upper Riemann Sum $U(P, f)$ for $f(x) = x^2$ and the partition $P = \{-1, 0, 2\}$ of the interval $[-1, 2]$.

Solution From the figure below, the maximum value of $f(x)$ on the subinterval $[-1, 0]$ occurs at $x = -1$ while the maximum value of $f(x)$ on the subinterval $[0, 2]$ occurs at $x = 2$. Therefore

$$U(P, f) \;=\; f(-1)\,[0-(-1)] + f(2)\,[2-0] \;=\; (1)(1) + (4)(2) \;=\; 9\ .$$



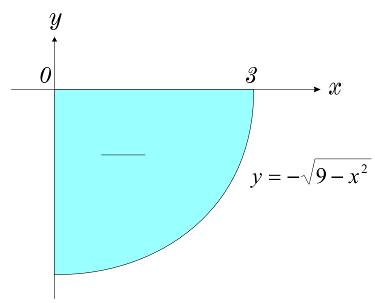
19. [2 Marks] Evaluate $\int_{-\pi/4}^{\pi/4} \tan x \; dx$.

Solution Since $y = \tan x$ is an odd function and the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is symmetric with respect to 0, $\int_{-\pi/4}^{\pi/4} \tan x \; dx = 0$.

20. [2 Marks] Evaluate $\int_0^3 -\sqrt{9-x^2}\; dx$.

Solution This integral represents the area of the quarter circle of radius 3 with center the origin depicted below. Since this region lies below the x–axis the definite integral gives the negative of its area. Thus

$$\int_0^3 -\sqrt{9-x^2}\; dx \;=\; -\frac{1}{4}\left[\pi(3)^2\right] \;=\; -\frac{9}{4}\pi\ .$$



21. [2 Marks] State the First Fundamental Theorem of Calculus.

Solution Let f be a continuous function with domain $[a, b]$. Define $F(x) = \int_a^x f(t)\ dt$ for $x \in [a, b]$. Then $F'(x) = f(x)$ for $x \in (a, b)$, $F'_+(a) = f(a)$ and $F'_-(b) = f(b)$.

22. [2 Marks] Find

$$\frac{d}{dx}\left(\int_1^{x^3} \text{arcsec}\ t\ dt\right) .$$

Solution By the First Fundamental Theorem of Calculus and the chain rule:

$$\frac{d}{dx}\left(\int_1^{x^3} \text{arcsec}\ t\ dt\right) = \text{arcsec}\ (x^3)\ \frac{d}{dx}(x^3) = 3x^2\text{arcsec}\ (x^3) .$$

23. [2 Marks] Observe that $D(x \sin x) = \sin x + x \cos x$. Evaluate

$$\int_0^\pi x \cos x\ dx.$$

Solution Since $D(\cos x) = -\sin x$, it follows that $D(x \sin x + \cos x) = x \cos x$. Hence

$$\int_0^\pi x \cos x\ dx = x \sin x + \cos x|_0^\pi = [0 + (-1)] - [0 + 1] = -2 .$$

A. [16 Marks] Consider the function

$$f(x) = \frac{x^{4/5}}{x+1} .$$

(a) Find the x–intercepts of f.

Solution $f(x) = 0$ when its numerator $x^{4/5} = 0$, i.e. when $x = 0$.

(b) Find the y–intercept of f.

Solution The y–intercept is $f(0) = 0$.

(c) Find the vertical asymptotes of f.

Solution The vertical asymptotes of f occur where its denominator is zero: $x+1=0$, i.e. $x=-1$.

(d) Find the horizontal asymptotes of f.

Solution The horizontal asymptotes of f are the limits of f as x goes to $+\infty$ and to $-\infty$:

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x^{4/5}}{x+1}\cdot\frac{1/x}{1/x} = \lim_{x\to\infty}\frac{1/x^{1/5}}{1+1/x} = \frac{0}{1+0} = 0\\
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{x^{4/5}}{x+1}\cdot\frac{1/x}{1/x} = \lim_{x\to-\infty}\frac{1/x^{1/5}}{1+1/x} = \frac{0}{1+0} = 0
\end{aligned}$$

Hence f has the x–axis ($y=0$) as a horizontal asymptote on both the left and the right.

(e) Determine where f is increasing and where f is decreasing.

Solution We determine where $f'(x)$ is positive and where it is negative. By the quotient rule

$$f'(x) = \frac{\frac{4}{5}x^{-1/5}(x+1)-x^{4/5}(1)}{(x+1)^2}\cdot\frac{5x^{1/5}}{5x^{1/5}} = \frac{4(x+1)-5x}{5x^{1/5}(x+1)^2} = \frac{4-x}{5x^{1/5}(x+1)^2}\,.$$

Note that $f'(x)$ changes sign at $x=0$ and at $x=4$. See the number line below. It follows that $f'(x)>0$ and f is increasing for $x\in(0,4)$ while $f'(x)<0$ and f is decreasing for $x\in(-\infty,0)\cup(4,\infty)$.

- - - - - DNE + + + + 0 - - - - - - sign f'(x)

$\longrightarrow x$

decreasing 0 increasing 4 decreasing

(f) Locate and identify the critical points of f.

Solution f has a critical point at $x=0$ where $f'(0)$ does not exist as well as at $x=4$ where $f'(4)=0$. By the First Derivative Test, f has a local minimum at $x=0$ and a local maximum at $x=4$.

(g) Determine where f is concave up and where f is concave down.

Solution We determine where $f''(x)$ is positive and where it is negative. By the quotient rule:

$$\begin{aligned}
f''(x) &= \frac{D(4-x)\left[x^{1/5}(x+1)^2\right]-(4-x)D\left[x^{1/5}(x+1)^2\right]}{5x^{2/5}(x+1)^4}\\
&= \frac{(-1)\left[x^{1/5}(x+1)^2\right]-(4-x)\left[\frac{1}{5}x^{-4/5}(x+1)^2+x^{1/5}2(x+1)\right]}{5x^{2/5}(x+1)^4}\\
&= \frac{-x^{1/5}(x+1)-(4-x)\left[\frac{1}{5}x^{-4/5}(x+1)+2x^{1/5}\right]}{5x^{2/5}(x+1)^3}\cdot\frac{5x^{4/5}}{5x^{4/5}}\\
&= \frac{-5x(x+1)-(4-x)\left[(x+1)+10x\right]}{25x^{6/5}(x+1)^3} = \frac{6x^2-48x-4}{25x^{6/5}(x+1)^3} = \frac{2(3x^2-24x-2)}{25x^{6/5}(x+1)^3}
\end{aligned}$$

$f''(x)$ changes sign at $x=-1$ and where its denominator is zero. By the quadratic formula, the numerator is zero when

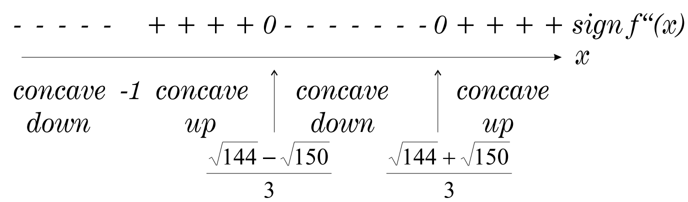
$$x = \frac{24\pm\sqrt{24^2-4(3)(-2)}}{6} = \frac{24\pm\sqrt{24^2+24}}{6} = \frac{12\pm\sqrt{150}}{3} = \frac{\sqrt{144}\pm\sqrt{150}}{3}$$

The sign of $f''(x)$ is given on the number line below. It follows that f is concave up for

$$x \in \left(-1, \frac{\sqrt{144}-\sqrt{150}}{3}\right) \cup \left(\frac{\sqrt{144}+\sqrt{150}}{3}, \infty\right)$$

while f is concave down for

$$x \in (-\infty, -1) \cup \left(\frac{\sqrt{144}-\sqrt{150}}{3}, \frac{\sqrt{144}+\sqrt{150}}{3}\right).$$

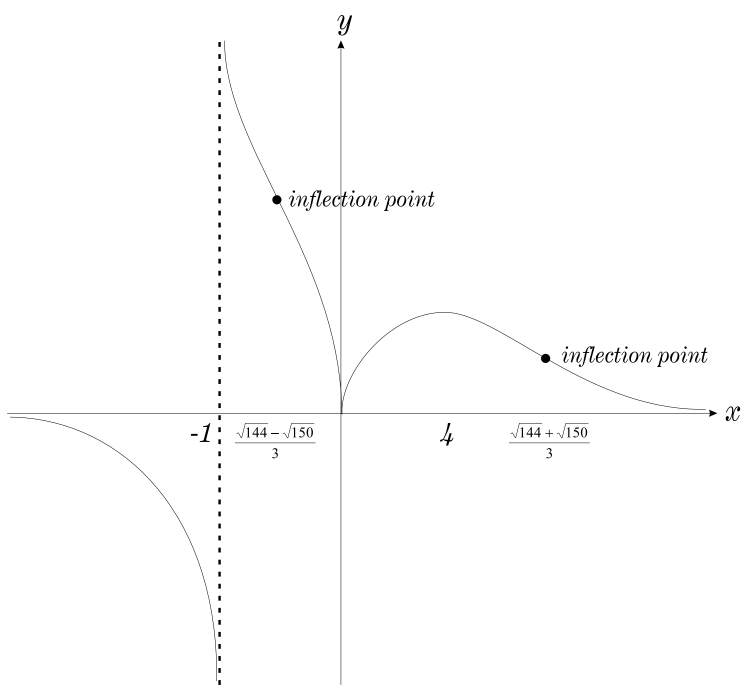


(h) Find the inflection points of f.

Solution f has inflection points at $x = \frac{\sqrt{144}\pm\sqrt{150}}{3}$ where $f''(x)$ changes sign. (Note that $x = -1$ is not an inflection point becuase $x = -1$ is not in the domain of f.)

(i) Sketch the graph of f. Indicate the inflection points on your graph.

Solution



B. [8 Marks] Use a number line to sketch the motion of an object whose position at time t, for all $t \in \Re$, is given by:

$$s(t) \;=\; t^6 - 3t^4 + 3t^2 \;.$$

That is, indicate where the particle moves left, where it moves right, where it speeds up, where it slows down and where it stops.

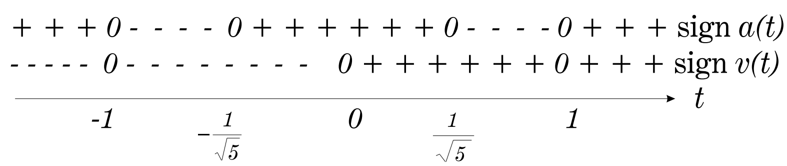
Solution The velocity is given by:

$$v(t) = s'(t) = 6t^5 - 12t^3 + 6t = 6t(t^4 - 2t^2 + 1) = 6t(t^2-1)^2 = 6t(t-1)^2(t+1)^2 \;.$$

Note that the velocity is zero for $t = -1$, $t = 0$ and $t = 1$. The acceleration is given by:

$$a(t) = v'(t) = 30t^4 - 36t^2 + 6 = 6(5t^4 - 6t^2 + 1) = 6(5t^2-1)(t^2-1) = 6(\sqrt{5}\,t-1)(\sqrt{5}\,t+1)(t-1)(t+1) \;.$$

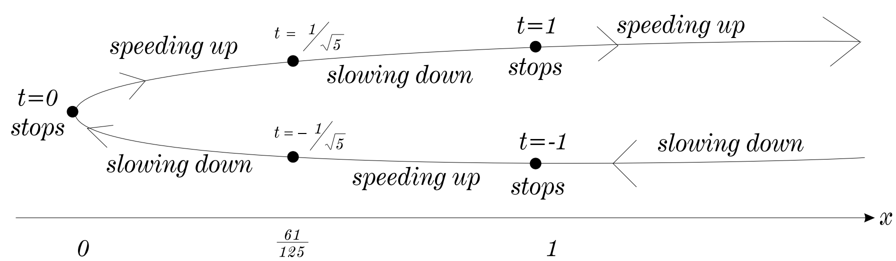
Note that the acceleration is zero for $t = -1$, $t = -1/\sqrt{5}$, $t = 1/\sqrt{5}$ and $t = 1$. The following diagram gives the signs of the velocity and acceleration.



Note that

$$\begin{aligned} s(0) &= 0 \\ s(1) &= s(-1) \;=\; 1 \\ s\left(\frac{1}{\sqrt{5}}\right) &= s\left(-\frac{1}{\sqrt{5}}\right) \;=\; \frac{1}{5^3} - \frac{3}{5^2} + \frac{3}{5} \;=\; \frac{1}{125}(1 - 15 + 75) = \frac{61}{125} \end{aligned}$$

Now we can sketch the motion of this object.



C. [8 Marks] A man two meters tall walks towards a five meter high street lamp at 6 km/hr. Find the rate that the shadow of the man on the ground is decreasing when the man is 20 meters from the lamp.

Solution Let s denote the length of the man's shadow on the ground, and let x denote the distance form the man to the lamp, both measured in meters. See the diagram below. Let t denote time measured in seconds. We are given that

$$\frac{dx}{dt} = -6 \text{ km/hr} = -6\,\frac{1000 \text{ m}}{3600 \text{ sec}} = -\frac{5}{3} \text{ m/sec} \;.$$

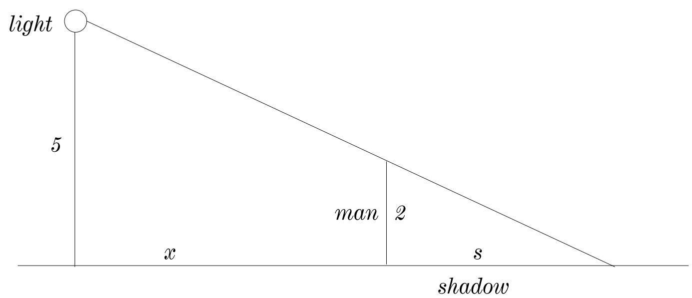
We are asked to find the value of $\frac{ds}{dt}$ when $x = 20$ meters. The large and small triangles in the figure below are similar. Therefore

$$\begin{aligned} \frac{s}{s+x} &= \frac{2}{5} \\ 5s &= 2s + 2x \\ 3s &= 2x \end{aligned}$$

Take the derivative of this equation with respect to t:

$$3\frac{ds}{dt} = 2\frac{dx}{dt} = 2\left(-\frac{5}{3}\right) = -\frac{10}{3} \text{ m/sec} .$$

Hence the length of the man's shadow is decreasing at $-\frac{10}{9}$ meters per second.



D. [8 Marks] Find the area A between the two curves $y = x^4 - x^2 - x + 2$ and $y = 4x^2 - x - 2$.

Solution Observe that these two curves intersect when

$$\begin{aligned} x^4 - x^2 - x + 2 &= 4x^2 - x - 2 \\ 0 = x^4 - 5x^2 + 4 &= (x^2-4)(x^2-1) = (x-2)(x+2)(x-1)(x+1), \end{aligned}$$

i.e. when $x = -2$, $x = -1$, $x = 1$ and $x = 2$. From the sketch of these curves below, we see that the area of the region between them is:

$$\begin{aligned} A &= \int_{-2}^{-1} (4x^2 - x - 2) - (x^4 - x^2 - x + 2)\, dx + \int_{-1}^{1} (x^4 - x^2 - x + 2) - (4x^2 - x - 2)\, dx \\ &\qquad + \int_{1}^{2} (4x^2 - x - 2) - (x^4 - x^2 - x + 2)\, dx \\ &= \int_{-2}^{-1} -x^4 + 5x^2 - 4\, dx + \int_{-1}^{1} x^4 - 5x^2 + 4\, dx + \int_{1}^{2} -x^4 + 5x^2 - 4\, dx \\ &= 2\int_{0}^{1} x^4 - 5x^2 + 4\, dx + 2\int_{1}^{2} -x^4 + 5x^2 - 4\, dx \end{aligned}$$

because $x^4 - 5x^2 + 4$ is an even function. Thus

$$A = 2\left(\frac{x^5}{5} - \frac{5x^3}{3} + 4x\Big|_0^1\right) + 2\left(-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\Big|_1^2\right)$$

$$= \quad 2\left(\frac{1}{5}-\frac{5}{3}+4\right) \;-\; 0 \;+\; 2\left(-\frac{32}{5}+\frac{40}{3}-8\right) \;-\; 2\left(-\frac{1}{5}+\frac{5}{3}-4\right)$$

$$= \quad 2\left(-\frac{30}{5}+\frac{30}{3}\right) \;=\; 2(-6+10) \;=\; 8\ .$$