SOLUTIONS TO FINAL EXAM

1. [10 Marks] Evaluate each of the following limits.

(a) $\lim_{x\to\infty}\left(1+\frac{2}{\ln x}\right)^{\ln x}$

Solution Let $u=\ln x$. As x approaches ∞, u also approaches ∞. Therefore

$$\lim_{x\to\infty}\left(1+\frac{2}{\ln x}\right)^{\ln x}=\lim_{u\to\infty}\left(1+\frac{2}{u}\right)^{u}=e^2.$$

(b) $\lim_{x\to 0}\frac{\sin x - x\cos x}{x-\arctan x}$

Solution Substitute the Taylor series at $x=0$ for $\sin x$, $\cos x$ and $\arctan x$. Then divide the numerator and denominator of the resulting fraction by x^3.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin x - x\cos x}{x-\arctan x} &= \lim_{x\to 0}\frac{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}-x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}}{x-\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}}\\
&= \lim_{x\to 0}\frac{\left(x-\frac{x^3}{6}+\sum_{n=2}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)-\left(x-\frac{x^3}{2}+x\sum_{n=2}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\right)}{x-\left(x-\frac{x^3}{3}+\sum_{n=2}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}\right)}\\
&= \lim_{x\to 0}\frac{\frac{x^3}{3}+x^5\sum_{n=2}^{\infty}(-1)^n\frac{x^{2n-4}}{(2n+1)!}-x^5\sum_{n=2}^{\infty}(-1)^n\frac{x^{2n-4}}{(2n)!}}{\frac{x^3}{3}-x^5\sum_{n=2}^{\infty}(-1)^n\frac{x^{2n-4}}{2n+1}}\\
&= \lim_{x\to 0}\frac{\frac{1}{3}+x^2\sum_{n=2}^{\infty}(-1)^n\frac{x^{2n-4}}{(2n+1)!}-x^2\sum_{n=2}^{\infty}(-1)^n\frac{x^{2n-4}}{(2n)!}}{\frac{1}{3}-x^2\sum_{n=2}^{\infty}(-1)^n\frac{x^{2n-4}}{2n+1}}=\frac{1/3}{1/3}=1
\end{aligned}$$

2. [25 Marks] Evaluate each of the following integrals.

(a) $\int\frac{e^x}{1+e^x}\,dx$

Solution

$$\int\frac{e^x}{1+e^x}\,dx=\int\frac{D(1+e^x)}{1+e^x}\,dx=\ln(1+e^x)+C$$

(b) $\int\frac{2x+3}{(x^2+2x+5)^2}\,dx$

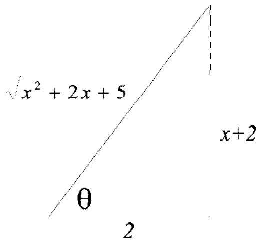
Solution Note $x^2+2x+5=(x+1)^2+2^2$. Hence make the trigonometric substiution $x+1=2\tan\theta$. Then $x=2\tan\theta-1$, $dx=2\sec^2\theta\ d\theta$ and

$$\begin{aligned}
\int\frac{2x+3}{(x^2+2x+5)^2}\,dx &= \int\frac{2(2\tan\theta-1)+3}{\left(4\tan^2\theta+4\right)^2}\,2\sec^2\theta\ d\theta=\int\frac{4\tan\theta+1}{\left(4\sec^2\theta\right)^2}\,2\sec^2\theta\ d\theta\\
&= \int\frac{4\tan\theta+1}{8\sec^2\theta}\,d\theta=\frac{1}{2}\int\frac{\tan\theta}{\sec^2\theta}\frac{\cos^2\theta}{\cos^2\theta}\,d\theta+\frac{1}{8}\int\frac{1}{\sec^2\theta}\,d\theta\\
&= \frac{1}{2}\int\sin\theta\cos\theta\ d\theta+\frac{1}{8}\int\cos^2\theta\ d\theta\\
&= \frac{1}{2}\int\frac{1}{2}\sin 2\theta\ d\theta+\frac{1}{8}\int\frac{1}{2}(1+\cos 2\theta)\ d\theta\\
&= -\frac{1}{4}\frac{1}{2}\cos 2\theta+\frac{1}{16}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C
\end{aligned}$$

$$\begin{aligned} &= -\frac{1}{8}\left(2\cos^2\theta - 1\right) + \frac{1}{16}(\theta + \sin\theta\cos\theta) + C \\ &= -\frac{1}{4}\cos^2\theta + \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + C \end{aligned}$$

Since $\tan\theta = \frac{x+1}{2}$, it follows that $\theta = \arctan\frac{x+1}{2}$. From the triangle below, $\sin\theta = \frac{x+1}{\sqrt{x^2+2x+5}}$ and $\cos\theta = \frac{2}{\sqrt{x^2+2x+5}}$. Thus

$$\begin{aligned} \int \frac{2x+3}{\left(x^2+2x+5\right)^2}\,dx &= -\frac{1}{4}\left(\frac{2}{\sqrt{x^2+2x+5}}\right)^2 + \frac{1}{16}\arctan\frac{x+1}{2} + \frac{1}{16}\left(\frac{x+1}{\sqrt{x^2+2x+5}}\right)\left(\frac{2}{\sqrt{x^2+2x+5}}\right) + C \\ &= \frac{x-7}{8(x^2+2x+5)} + \frac{1}{16}\arctan\frac{x+1}{2} + C \end{aligned}$$



(c) $\displaystyle\int 2^x \sin x\,dx$

Solution Use integration by parts twice.

$$\begin{aligned} \int 2^x \sin x\,dx &= \int 2^x D(-\cos x)\,dx = 2^x(-\cos x) - \int D(2^x)(-\cos x)\,dx \\ &= -2^x\cos x + \ln 2\int 2^x\cos x\,dx = -2^x\cos x + \ln 2\int 2^x D(\sin x)\,dx \\ &= -2^x\cos x + (\ln 2)2^x\sin x - \ln 2\int D(2^x)\sin x\,dx \\ &= -2^x\cos x + (\ln 2)2^x\sin x - (\ln 2)^2\int 2^x\sin x\,dx \\ \left[1+(\ln 2)^2\right]\int 2^x\sin x\,dx &= -2^x\cos x + (\ln 2)2^x\sin x + C \\ \int 2^x\sin x\,dx &= -\frac{1}{1+(\ln 2)^2}2^x\cos x + \frac{\ln 2}{1+(\ln 2)^2}2^x\sin x + C \end{aligned}$$

(d) $\displaystyle\int \sec^6 x\,dx$

Solution Since the exponent of $\sec x$ is even, let $u = \tan x$ with $du = \sec^2 x\,dx$:

$$\begin{aligned} \int \sec^6 x\,dx &= \int \left(\sec^2 x\right)^2 \sec^2 x\,dx = \int \left(1+\tan^2 x\right)^2\sec^2 x\,dx = \int \left(1+u^2\right)^2\,du \\ &= \int 1 + 2u^2 + u^4\,du = u + \frac{2}{3}u^3 + \frac{1}{5}u^5 + C \\ &= \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C \end{aligned}$$

(e) $\displaystyle\int \frac{7x^3+6x^2+x-2}{(x+1)^2(x^2+1)}\,dx$

Solution We write the integrand as:

$$\begin{aligned}\frac{7x^3+6x^2+x-2}{(x+1)^2(x^2+1)} &= \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+1}\\ &= \frac{A(x+1)(x^2+1)+B(x^2+1)+(Cx+D)(x+1)^2}{(x+1)^2(x^2+1)}\end{aligned}$$

Multiply this equation by $(x+1)^2(x^2+1)$:

$$7x^3+6x^2+x-2=A(x+1)(x^2+1)+B(x^2+1)+(Cx+D)(x+1)^2$$

Let $x=-1$: $-7+6-1-2=B(2)$, so $-4=2B$ and $B=-2$.
Substitute this value of B into the previous equation.

$$\begin{aligned}7x^3+6x^2+x-2 &= A(x+1)(x^2+1)-2(x^2+1)+(Cx+D)(x+1)^2\\ 7x^3+6x^2+x-2+2(x^2+1) &= A(x+1)(x^2+1)+(Cx+D)(x+1)^2\\ 7x^3+8x^2+x &= A(x+1)(x^2+1)+(Cx+D)(x+1)^2\\ (x+1)(7x^2+x) &= (x+1)\left[A(x^2+1)+(Cx+D)(x+1)\right]\\ 7x^2+x &= A(x^2+1)+(Cx+D)(x+1)\end{aligned}$$

Let $x=-1$: $7-1=A(2)$, so $6=2A$ and $A=3$.
Substitute this value of A into the previous equation.

$$\begin{aligned}7x^2+x &= 3(x^2+1)+(Cx+D)(x+1)\\ 7x^2+x-3(x^2+1) &= (Cx+D)(x+1)\\ 4x^2+x-3 &= (Cx+D)(x+1)\\ (x+1)(4x-3) &= (Cx+D)(x+1)\\ 4x-3 &= Cx+D\end{aligned}$$

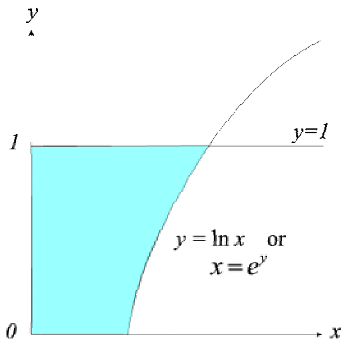
Thus $C=4$ and $D=-3$. Now

$$\begin{aligned}\int\frac{7x^3+6x^2+x-2}{(x+1)^2(x^2+1)}\,dx &= \int\frac{3}{x+1}+\frac{-2}{(x+1)^2}+\frac{4x-3}{x^2+1}\,dx\\ &= 3\ln|x+1|+\frac{2}{x+1}+2\int\frac{2x}{x^2+1}\,dx-3\int\frac{1}{1+x^2}\,dx\\ &= 3\ln|x+1|+\frac{2}{x+1}+2\ln(x^2+1)-3\arctan x+C\end{aligned}$$

3. [4 Marks] Let R be the region in the first quadrant of the xy–plane bounded by the graph of $y=\ln x$, the line $y=1$, the x–axis and the y–axis. Let S be the solid obtained by revolving the region R around the y–axis. Find the volume V of S.

Solution Observe that the graph of $y=\ln x$ is also the graph of $x=e^y$. From the sketch of the region R below,

$$\begin{aligned}V &= \int_0^1\pi\left(e^y\right)^2\,dy=\pi\int_0^1 e^{2y}\,dy\\ &= \frac{\pi}{2}e^{2y}\Big|_0^1=\frac{\pi}{2}e^2-\frac{\pi}{2}e^0=\frac{\pi}{2}\left(e^2-1\right)\end{aligned}$$

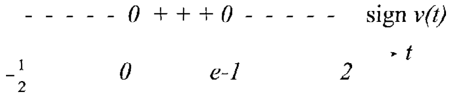


4. [4 Marks] Find the distance D traveled by an object from $t = -\frac{1}{2}$ sec to $t = 2$ sec with velocity

$$v(t) = \left[e^t - 1\right] \; \left[1 - \ln(1+t)\right] \quad \text{meters/sec.}$$

WRITE YOUR ANSWER AS AN INTEGRAL (OR SUM OF INTEGRALS).
DO NOT INTEGRATE.

Solution Note the increasing function e^t equals one when $t = 0$. Hence $e^t - 1 < 0$ for $t < 0$ and $e^t - 1 > 0$ for $t > 0$.
Also the increasing function $\ln(1+t)$ equals one when $1 + t = e$, i.e. $t = e - 1$. Hence
$1 - \ln(1+t) > 0$ for $-\frac{1}{2} \leq t < e - 1$ and $1 - \ln(1+t) < 0$ for $e - 1 < t \leq 2$. The sign of $v(t)$ is given by the following figure.



Therefore

$$\begin{aligned} D &= \int_{1/2}^{2} |v(t)| \; dt = \int_{1/2}^{0} -v(t) \; dt + \int_{0}^{e-1} v(t) \; dt + \int_{e-1}^{2} -v(t) \; dt \\ &= \int_{1/2}^{0} \left[e^t - 1\right] \; \left[\ln(1+t) - 1\right] \; dt + \int_{0}^{e-1} \left[e^t - 1\right] \; \left[1 - \ln(1+t)\right] \; dt \\ & \qquad + \int_{e-1}^{2} \left[e^t - 1\right] \; \left[\ln(1+t) - 1\right] \; dt \;\; \text{meters} \end{aligned}$$

5. [4 Marks] Estimate $\ln 3$ with error less than 0.1
JUSTIFY YOUR ANSWER.

Solution $\ln 3 = \int_1^3 \frac{1}{x} \; dx$. We approximate this integral by Simpson's Rule using the regular partition P_{2m} of $[1, 3]$ into $2m$ subintervals of length $\Delta x = \frac{2}{2m} = \frac{1}{m}$. The error in this approximation is less than $\frac{(b-a)^5 M}{180 n^4}$ where $b = 3$, $a = 1$, $n = 2m$ and M is an upper bound of the absolute value of the fourth derivative of $f(x) = \frac{1}{x}$ for $x \in [1, 3]$. Now

$$f'(x) = -\frac{1}{x^2} \quad f^{(2)}(x) = \frac{2}{x^3} \quad f^{(3)}(x) = -\frac{6}{x^4} \quad f^{(4)}(x) = \frac{24}{x^5}$$

Hence $M = 24$. Thus the error in Simposon's Rule using P_{2m} is:

$$\frac{(b-a)^5 M}{180 n^4} = \frac{(3-1)^5 24}{180(2^4) m^4} = \frac{2^5(2)}{15(2^4) m^4} = \frac{4}{15 m^4}$$

We want $\frac{4}{15m^4} < 0.1 = \frac{1}{10}$, i.e. $15m^4 > 40$ or $m^4 > \frac{40}{15} = \frac{8}{3}$. Thus take $m = 2$, i.e $2m = 4$. By Simpson's Rule

$$\ln 3 \;\; \approx \;\; \frac{\Delta x}{3} \left[f(1) + 4f(3/2) + 2f(2) + 4f(5/2) + f(3)\right]$$

$$\begin{aligned}
\ln 3 &\approx \frac{1/2}{3}\left[1+4\frac{2}{3}+2\frac{1}{2}+4\frac{2}{5}+\frac{1}{3}\right] \\
\ln 3 &\approx \frac{1}{6}\left[1+\frac{8}{3}+1+\frac{8}{5}+\frac{1}{3}\right]=\frac{1}{6}\left[5+\frac{8}{5}\right]=\frac{1}{6}\cdot\frac{33}{5} \\
\ln 3 &\approx \frac{11}{10}=1.1
\end{aligned}$$

6. [10 Marks] Determine whether each improper intgeral converges or diverges.
JUSTIFY YOUR ANSWERS.

(a) $\displaystyle\int_0^1 \ln x\ dx.$

Solution Note that $y=\ln x$ has a vertical asymptote on the left at $x=0$. By the definition of this improper integral and integration by parts:

$$\begin{aligned}
\int_0^1 \ln x\ dx &= \lim_{a\to 0^+}\int_a^1 \ln x\ dx = \lim_{a\to 0^+}\int_a^1 D(x)\ln x\ dx \\
&= \lim_{a\to 0^+}\left(x\ln x|_a^1-\int_a^1 xD(\ln x)\ dx\right)=\lim_{a\to 0^+}\left(\ln 1-a\ln a-\int_a^1 x\frac{1}{x}\ dx\right) \\
&= \lim_{a\to 0^+}\left(-a\ln a-\int_a^1 1\ dx\right)=\lim_{a\to 0^+}\left(-a\ln a-x|_a^1\right) \\
&= \lim_{a\to 0^+}\left[-a\ln a-(1-a)\right]=-1-\lim_{a\to 0^+}\frac{\ln a}{1/a}
\end{aligned}$$

The latter fraction when $a=0^+$ becomes $\frac{-\infty}{\infty}$. Hence apply L'Hôpital's Rule:

$$\begin{aligned}
\int_0^1 \ln x\ dx = 1-\lim_{a\to 0^+}\frac{D(\ln a)}{D(1/a)} &= 1-\lim_{a\to 0^+}\frac{1/a}{-1/a^2} \\
&= 1+\lim_{a\to 0^+} a = 1
\end{aligned}$$

Thus this improper integral converges.

(b) $\displaystyle\int_{-\infty}^{\infty}\frac{\cos^2 x}{x^2}\ dx.$

Solution Note the integrand has the y–axis as a vertical asymptote. Hence

$$\int_{-\infty}^{\infty}\frac{\cos^2 x}{x^2}\ dx=\int_{-\infty}^{-1}\frac{\cos^2 x}{x^2}\ dx+\int_{-1}^{0}\frac{\cos^2 x}{x^2}\ dx+\int_{0}^{1}\frac{\cos^2 x}{x^2}\ dx+\int_{1}^{\infty}\frac{\cos^2 x}{x^2}\ dx$$

For $0\le x\le 1<\frac{\pi}{3}$, the decreasing function $y=\cos x$ satisfies
$\frac{1}{2}=\cos\frac{\pi}{3}\le\cos x\le 1$. Hence for $0\le x\le 1$:

$$\frac{\cos^2 x}{x^2}\ge\frac{1/2}{x^2}$$

Since $2\ge 1$, we know $\int_0^1\frac{1/2}{x^2}\ dx=\frac{1}{2}\int_0^1\frac{1}{x^2}\ dx$ diverges. By the Comparison Test, $\int_0^1\frac{\cos^2 x}{x^2}\ dx$ also diverges.
Hence $\int_{-\infty}^{\infty}\frac{\cos^2 x}{x^2}\ dx$ diverges.

7. [5 Marks] Find the general solution of the differential equation

$$y'=2xy+2x+y+1.$$

Write your answer as y equals a specific function of x.
JUSTIFY YOUR ANSWER.

Solution Observe that $2xy + 2x + y + 1 = (2x+1)(y+1)$. Hence for $y \neq -1$:

$$\begin{aligned}
\frac{dy}{dx} &= (2x+1)(y+1) \\
\frac{1}{y+1}\frac{dy}{dx} &= 2x+1 \\
\int \frac{1}{y+1}\frac{dy}{dx}\,dx &= \int 2x+1\,dx \\
\int \frac{1}{y+1}\,dy &= x^2+x+C \\
\ln|y+1| &= x^2+x+C \\
|y+1| &= \exp(x^2+x+C) = e^c\exp(x^2+x) \\
y+1 &= \pm e^c\exp(x^2+x) = A\exp(x^2+x) \\
y &= A\exp(x^2+x) - 1
\end{aligned}$$

for all nonzero numbers A. Moreover, $y = -1$ is also a solution. This is the case $A = 0$. Thus the general solution is:

$$y = A\exp(x^2+x) - 1 \quad \text{for } A \in \Re.$$

8. [8 Marks] Determine whether each sequence is increasing or decreasing; bounded below or bounded above. JUSTIFY YOUR ANSWERS.

(a) $\left\{\dfrac{(2n+1)!}{(n!)^2}\right\}$ for $n \geq 1$.

Solution Consider the $(n+1)^{\text{st}}$ term a_{n+1} divided by the n^{th} term a_n:

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\frac{[2(n+1)+1]!}{[(n+1)!]^2}}{\frac{(2n+1)!}{(n!)^2}} = \frac{(2n+3)!}{(2n+1)!}\left(\frac{n!}{(n+1)!}\right)^2 = \frac{(2n+3)(2n+2)(2n+1)!}{(2n+1)!}\left(\frac{n!}{(n+1)n!}\right)^2 \\
&= (2n+3)(2n+2)\left(\frac{1}{n+1}\right)^2 = 2(2n+3)(n+1)\frac{1}{(n+1)^2} = \frac{4n+6}{n+1} \\
&= 4 + \frac{2}{n+1} > 4.
\end{aligned}$$

Hence the given sequence is increasing. Therefore it is bounded below by its first term $a_1 = \frac{3!}{(1!)^2} = 6$. Since each term is at least four times as large as the preceding one, the limit of this sequence is ∞. Therefore, this sequence is not bounded above.

(b) $\left\{\arcsin\dfrac{1}{n}\right\}$ for $n \geq 1$.

Solution Let $f(x) = \arcsin\frac{1}{x}$ for $x \geq 1$. By the chain rule:

$$f'(x) = \frac{1}{\sqrt{1-1/x^2}}\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{\sqrt{1-1/x^2}}\left(-\frac{1}{x^2}\right) < 0$$

Hence f is a decreasing function, and the given sequence is decreasing. Therefore it is bounded above by its first term $\arcsin 1 = \frac{\pi}{2}$. Since it is a sequence of positive numbers it is bounded below by 0.

9. [8 Marks] Determine whether each of the following sequences converges or diverges. If it converges, find its limit. JUSTIFY YOUR ANSWERS.

(a) $\left\{(-1)^n\left(1+\dfrac{1}{n}\right)^n\right\}$ for $n \geq 1$

Solution Recall $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. Hence the even terms of the given sequence have limit e while the odd terms have limit $-e$. Therefore, the given sequence diverges.

(b) $\left\{(-1)^n\left(1-\dfrac{1}{n}\right)^{n^2}\right\}$ for $n \geq 1$

Solution Let $L = \lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{n^2}$. We compute $\ln L$ by appplying L'Hôpital's Rule to the indeterminate form $\frac{0}{0}$:

$$\begin{aligned}
\ln L &= \ln\left[\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n^2}\right] = \lim_{n\to\infty}\ln\left[\left(1-\frac{1}{n}\right)^{n^2}\right] \\
&= \lim_{n\to\infty} n^2 \ln\left(1-\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1-\frac{1}{n}\right)}{1/n^2} = \lim_{n\to\infty}\frac{D\left[\ln\left(1-\frac{1}{n}\right)\right]}{D(1/n^2)} \\
&= \lim_{n\to\infty}\frac{1/\left(1-\frac{1}{n}\right)\left(1/n^2\right)}{-2/n^3} = -\frac{1}{2}\lim_{n\to\infty}\frac{n}{1-\frac{1}{n}} = -\infty
\end{aligned}$$

Thus $L = e^{\ln L} = 0$. Note that

$$\begin{aligned}
-\left(1-\frac{1}{n}\right)^{n^2} &\leq (-1)^n\left(1-\frac{1}{n}\right)^{n^2} \leq \left(1-\frac{1}{n}\right)^{n^2} \\
\lim_{n\to\infty} -\left(1-\frac{1}{n}\right)^{n^2} &= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n^2} = 0.
\end{aligned}$$

By the Pinching Theorem, $\lim\limits_{n\to\infty}(-1)^n\left(1-\dfrac{1}{n}\right)^{n^2} = 0.$

10. [12 Marks] Determine whether each series is absolutely convergent, conditionally convergent or divergent. JUSTIFY YOUR ANSWERS.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\, ne^{-n^2}.$

Solution We use the Integral Test to check for absolute convergence. Let $f(x) = xe^{-x^2}$ for $x \geq 1$. By the product rule:

$$f'(x) = (1)e^{-x^2} + xe^{-x^2}(-2x) = e^{-x^2}(1-2x^2) \leq 0$$

for $x \geq 1$. Thus f is a decreasing function, and $f(x) \geq 0$ for $x \geq 1$. Thus the Integral Test applies to the series $\sum_{n=1}^{\infty} ne^{-n^2}$.

$$\begin{aligned}
\int_1^{\infty} xe^{-x^2}\,dx &= \lim_{b\to\infty}\int_1^b xe^{-x^2}\,dx = \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^b \\
&= \lim_{b\to\infty} -\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-1} = 0 + \frac{1}{2e} = \frac{1}{2e}
\end{aligned}$$

Since this improper integral converges, the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges by the Integral Test. Hence the series $\sum_{n=1}^{\infty}(-1)^n\, ne^{-n^2}$ is absolutely convergent.

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\,\cos^2\frac{1}{n}.$

Solution Note

$$\lim_{n\to\infty}\cos^2\frac{1}{n} = \cos^2\left(\lim_{n\to\infty}\frac{1}{n}\right) = \cos^2 0 = 1.$$

Hence $\lim\limits_{n\to\infty}(-1)^n\cos^2\dfrac{1}{n}$ does not exist, and the series $\sum_{n=1}^{\infty}(-1)^n\,\cos^2\frac{1}{n}$ diverges by the n^{th} Term Test.

(c) $\sum_{n=1}^{\infty}(-1)^n \dfrac{\cos \pi n}{1+\sqrt{n}}$.

Solution Since $\cos \pi n = (-1)^n$,

$$\sum_{n=1}^{\infty}(-1)^n \frac{\cos \pi n}{1+\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$$

Apply the Limit Comparison Test to the latter series and the p–series with $p = 1/2$:

$$\lim_{n\to\infty} \frac{\frac{1}{1+\sqrt{n}}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1.$$

Since $1/2 \le 1$, the p-series with $p = 1/2$ diverges, i.e $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. By the Limit Comparison Test, the series $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ also diverges.

11. [5 Marks] Find the interval of convergence of the power series $\sum_{n=2}^{\infty} \dfrac{x^n}{\ln n}$.
JUSTIFY YOUR ANSWER.

Solution Apply the Ratio Test:

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{x^{n+1}/\left[\ln(n+1)\right]}{x^n/(\ln n)} \right| = \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \frac{|x|^{n+1}}{|x|^n} \\ &= \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}|x| \end{aligned}$$

This limit is the indeterminate form $\frac{\infty}{\infty}$, and we evaluate it by L'Hôpital's Rule:

$$L = \lim_{n\to\infty} \frac{D(\ln n)}{D\left[\ln(n+1)\right]}|x| = \lim_{n\to\infty} \frac{1/n}{1/(n+1)}|x| = \lim_{n\to\infty} \frac{n+1}{n}|x| = (1)|x| = |x|$$

Thus this power series converges when $L = |x| < 1$ ($-1 < x < 1$) and diverges for $L = |x| > 1$ ($x < -1$ or $x > 1$). When $x = 1$ the power series is:

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}.$$

This series is greater than the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ which diverges by the Integral Test. By the Comparison Test the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ also diverges. When $x = -1$ the power series is:

$$\sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln n}.$$

Note that $\left\{\frac{1}{\ln n}\right\}$ is a decreasing sequence of positive numbers with limit zero. Hence the series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln n}$ converges by the Alternating Series Test.
Thus the interval of convergence of this power series is $[-1, 1)$.

12. [5 Marks] Estimate $\int_0^1 \exp(-x^2)\, dx$ with error less than 0.01 Write your answer as a fraction.
JUSTIFY YOUR ANSWER.

Solution Recall that $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ for $u \in \Re$. Substitute $u = -x^2$:

$$\exp(-x^2) = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

for $x \in \Re$. Thus

$$\begin{aligned}
\int_0^1 \exp(-x^2)\ dx &= \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\ dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}\Bigg|_0^1 \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(2n+1)}
\end{aligned}$$

This is an alternating series. Therefore, the error in estimating this integral by the sum of the terms $n = 0, \ldots, t$ is less than than the absolute value of the summand $n = t + 1$, i.e. $\frac{1}{(t+1)!(2t+3)}$. Note

$$\frac{1}{2!(5)} = \frac{1}{10}, \qquad \frac{1}{3!(7)} = \frac{1}{42}, \qquad \frac{1}{4!(9)} = \frac{1}{216} < \frac{1}{100}.$$

Thus we take $t = 3$ to estimate this integral with error less than .01:

$$\begin{aligned}
\int_0^1 \exp(-x^2)\ dx &\approx \frac{1}{(0!)(1)} - \frac{1}{(1!)(3)} + \frac{1}{(2!)(5)} - \frac{1}{(3!)(7)} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} \\
\int_0^1 \exp(-x^2)\ dx &\approx \frac{210 - 70 + 21 - 5}{210} = \frac{156}{210} = \frac{26}{35}
\end{aligned}$$