

## SOLUTIONS TO EXAM 2

Part I: each question is worth 4 marks.

1. Find the equation of the surface of revolution obtained by rotating the graph of  $y = \frac{x^2}{1+x^2}$  around the  $x$ -axis.

**Solution** Replace  $y$  by  $\sqrt{y^2 + z^2}$  to obtain the equation of this surface:

$$\sqrt{y^2 + z^2} = \frac{x^2}{1 + x^2}.$$

2. Identify the quadric surface  $x^2 + 4x + y^2 - 2y = z^2 - 6z$ .

**Solution** Begin by completing the square:

$$\begin{aligned}(x + 2)^2 + (y - 1)^2 + 9 &= (z - 3)^2 + 4 + 1 \\ 4 &= (z - 3)^2 - (x + 2)^2 - (y - 1)^2\end{aligned}$$

This is a hyperboloid of two sheets with center  $(-2, 1, 3)$ .

3. Describe the level surfaces of the function  $w = \frac{x^2}{z+y^2}$ .

**Solution** The level surfaces are  $c = \frac{x^2}{z+y^2}$  or  $c(z + y^2) = x^2$ .

When  $c < 0$ , we have  $cz = x^2 - cy^2$  which is an elliptic paraboloid.

When  $c = 0$ , we have  $0 = x^2$  which is the plane  $x = 0$ .

When  $c > 0$ , we have  $cz = x^2 - cy^2$  which is a hyperbolic paraboloid.

4. Describe the domain of the function  $f(x, y) = \frac{xy}{x^2 - y^2}$ .

**Solution** The domain of  $f$  consists of all points  $(x, y) \in \mathbb{R}^2$  where the denominator  $x^2 - y^2 = (x - y)(x + y)$  is nonzero, i.e. all point which are neither on the line  $y = x$  nor on the line  $y = -x$ .

5. Evaluate  $\lim_{(x,y) \rightarrow (1,2)} \frac{xy - x - y + 1}{xy + 2x - y - 2}$ .

**Solution** Substitution of  $x = 1$ ,  $y = 2$  into this fraction produces  $\frac{0}{0}$ . Factor the numerator and denominator:

$$\lim_{(x,y) \rightarrow (1,2)} \frac{xy - x - y + 1}{xy + 2x - y - 2} = \lim_{(x,y) \rightarrow (1,2)} \frac{(x-1)(y-1)}{(x-1)(y+2)} = \lim_{(x,y) \rightarrow (1,2)} \frac{y-1}{y+2} = \frac{2-1}{2+2} = \frac{1}{4}.$$

**Problems 6 to 10: Consider the following five subsets of the coordinate plane  $\mathbb{R}^2$ .**

$A$  is the disc  $x^2 + y^2 < 1$ .

$B$  is the square of  $(x, y)$  with  $0 \leq x < 2$  and  $4 \leq y \leq 6$ .

$C$  is the annulus  $4 \leq x^2 + y^2 \leq 9$ .

$D$  is the set of  $(x, y)$  with  $x < 5$  or  $y \geq -2$ .

$E$  is the set of  $(x, y)$  with  $|x| \geq 3$ .

6. List each of these sets which is a closed region.

**Solution**  $C$  and  $E$  are the only closed regions.

7. List each of these sets which is an open region.

**Solution**  $A$  is the only open region.

8. List each of these sets which is path connected.

**Solution**  $A, B, C, D$  are path connected.

9. List each of these sets which is bounded.

**Solution**  $A, B$  and  $C$  are bounded.

10. Find the boundary of  $C$ .

**Solution** The boundary of  $C$  consists of the two circles  $x^2 + y^2 = 4$  and  $x^2 + y^2 = 9$ .

11. Let  $f(x, y) = 3x^2 - 5y^2$ ,  $\mathbf{c} = (-1, 2)$  and  $\mathbf{h} = (h_1, h_2)$ . Find  $E(\mathbf{c}; \mathbf{h})$ .  
SIMPLIFY YOUR ANSWER.

**Solution** Observe that  $\nabla f = (6x, -10y)$ .

$$\begin{aligned} E(\mathbf{c}, \mathbf{h}) &= f(\mathbf{c} + \mathbf{h}) - f(\mathbf{c}) - \nabla f(\mathbf{c}) \cdot \mathbf{h} = f(h_1 - 1, 2 + h_2) - f(-1, 2) - \nabla f(-1, 2) \cdot (h_1, h_2) \\ &= 3(h_1 - 1)^2 - 5(2 + h_2)^2 + 17 - (-6, -20) \cdot (h_1, h_2) \\ &= 3(h_1 - 1)^2 - 5(2 + h_2)^2 + 17 - (-6h_1 - 20h_2) \\ &= 3(h_1^2 - 2h_1 + 1) - 5(4 + 4h_2 + h_2^2) + 17 + 6h_1 + 20h_2 \\ &= 3h_1^2 - 6h_1 + 3 - (20 + 20h_2 + 5h_2^2) + 17 + 6h_1 + 20h_2 = 3h_1^2 - 5h_2^2 \end{aligned}$$

12. Let  $\nabla f(-2, 3, 1) = (5, -4, 2)$ . Find the equation of the tangent plane to the level surface of  $f$  through the point  $(-2, 3, 1)$ .

**Solution** The vector  $\nabla f(-2, 3, 1)$  is perpendicular to this level surface at  $(-2, 3, 1)$ . Hence the tangent plane has equation

$$\begin{aligned} 0 &= \nabla f(-2, 3, 1) \cdot (x + 2, y - 3, z - 1) = (5, -4, 2) \cdot (x + 2, y - 3, z - 1) = 5(x + 2) - 4(y - 3) + 2(z - 1) \\ 20 &= -5x + 4y - 2z \end{aligned}$$

13. Let  $\nabla g(4, -3) = (-2, 5)$ . Find the value of the smallest directional derivative of  $g$  at  $(4, -3)$ .

**Solution** The smallest directional derivative has value the negative of the length gradient. The gradient has length  $\sqrt{(-2)^2 + 5^2} = \sqrt{29}$ . Hence the value of the smallest directional derivative is  $-\sqrt{29}$ .

14. How do we know that the function  $f(x, y) = \sin(x^2 + y^2)$  is differentiable at every point of  $\mathfrak{R}^2$ ?

**Solution** Notice that  $f$  is a  $C^1$  function because its partial derivatives are continuous:

$$f_x = 2x \cos(x^2 + y^2), \quad f_y = 2y \cos(x^2 + y^2) .$$

Hence  $f$  is a differentiable function.

15. Let  $(x, y)$  be a point in the disc  $x^2 + y^2 \leq 4$ . Use a corollary of the Mean Value Theorem to explain why  $\sin^2(x + y) \leq 2(x^2 + y^2)$ .

**Solution** Observe that if  $f(x, y) = \sin(x + y)$ , then

$$\nabla f(x, y) = (\cos(x + y), \cos(x + y)) = \cos(x + y) \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Hence  $|\nabla f(x, y)| \leq (1)(\sqrt{2}) = \sqrt{2}$ . By a Corollary to the Mean Value Theorem:

$$\begin{aligned} |f(x, y) - f(0, 0)| &\leq \sqrt{2} |(x - 0, y - 0)| \\ |\sin(x + y) - \sin 0| &\leq \sqrt{2} \sqrt{x^2 + y^2} \end{aligned}$$

Square this equation:

$$\sin^2(x + y) \leq 2(x^2 + y^2) .$$

16. Let  $f(x, y) = xy$ ,  $r(t) = (t \cos t, t \sin t)$  and  $g(t) = f(r(t))$ . Find  $g'(t)$ .

**Solution** By the Chain Rule:

$$\begin{aligned} g'(t) &= \nabla f \cdot \vec{r}'(t) = (y, x) \cdot (\cos t - t \sin t, \sin t + t \cos t) = y(\cos t - t \sin t) + x(\sin t + t \cos t) \\ &= (t \sin t)(\cos t - t \sin t) + (t \cos t)(\sin t + t \cos t) = 2t \sin t \cos t - t^2 \sin^2 t + t^2 \cos^2 t \\ &= t \sin 2t + t^2 \cos 2t \end{aligned}$$

17. Let  $f(x, y)$ ,  $g(x, y)$  and  $h(x, y)$  be  $C^1$ -functions with domain  $\mathfrak{R}^2$ . Assume that  $g(-4, 5) = 2$ ,  $h(-4, 5) = -3$ ,  $\nabla f(2, -3) = (6, -1)$ ,  $\nabla g(-4, 5) = (-5, -2)$  and  $\nabla h(-4, 5) = (3, 7)$ . Let  $Q(x, y) = f(g(x, y), h(x, y))$ . Find  $\nabla Q(-4, 5)$ .

**Solution** We find the partial derivatives of  $Q$  by the Chain Rule:

$$\begin{aligned} \frac{\partial Q}{\partial x}(-4, 5) &= \nabla f(2, -3) \cdot (g_x(-4, 5), h_x(-4, 5)) = (6, -1) \cdot (-5, 3) = (6)(-5) + (-1)(3) = -33 \\ \frac{\partial Q}{\partial y}(-4, 5) &= \nabla f(2, -3) \cdot (g_y(-4, 5), h_y(-4, 5)) = (6, -1) \cdot (-2, 7) = (6)(-2) + (-1)(7) = -19 \end{aligned}$$

Hence  $\nabla Q(-4, 5) = (-33, -19)$ .

18. The equation  $x^2 z^3 = y^2 z + 3$  defines  $z$  as a function of  $x$  and  $y$ . Find  $\frac{\partial z}{\partial y}$  at the point  $(2, -1, 1)$ .

**Solution** The given equation can be written as  $f(x, y, z) = x^2 z^3 - y^2 z - 3 = 0$ . By implicit differentiation:

$$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = -\frac{-2yz}{3x^2 z^2 - y^2} = \frac{2yz}{3x^2 z^2 - y^2} .$$

When  $x = 2$ ,  $y = -1$ ,  $z = 1$ :  $\frac{\partial z}{\partial y} = \frac{2(-1)(1)}{3(4)(1) - 1} = -\frac{2}{11}$ .

19. Identify the stationary point  $(0, 0)$  of the function  $f(x, y) = x^2 + xy + y^2$ .

**Solution**  $f_x = 2x + y$  and  $f_y = x + 2y$ . Hence

$$A = f_{xx} = 2, \quad B = f_{xy} = 1, \quad C = f_{yy} = 2, \quad D = B^2 - AC = (1)^2 - (2)(2) = -3 .$$

Since  $D < 0$  and  $A > 0$ , the stationary point  $(0, 0)$  is a local minimum.

**20.** Give an example of a specific function  $z = f(x, y)$  and a specific point  $(a, b)$  where  $f$  has a stationary point which is not a local extremum.  
JUSTIFY YOUR ANSWER.

**Solution** Let  $f(x, y) = x^2 - y^2$  with  $(a, b) = (0, 0)$ .

$\nabla f = (2x, -2y)$ , so  $\nabla f(0, 0) = (0, 0)$ . Hence  $(0, 0)$  is a stationary point.

Apply the Second Derivative Test:

$A = f_{xx} = 2$ ,  $B = f_{xy} = 0$ ,  $C = f_{yy} = -2$  and  $D = B^2 - AC = 0 - (2)(-2) = 4$ .

Since  $D > 0$  at  $(0, 0)$ , the stationary point  $(0, 0)$  is a saddle point, hence is not a local extremum.

Part II: each question is worth 10 marks.

**A.** Find the sine of the angle between the surface  $z = x^2 + y^2$  and the curve  $r(t) = (\cos \pi t, \sin \pi t, t)$ .

**Solution** This surface intersects this curve when  $t = \cos^2 \pi t + \sin^2 \pi t = 1$ . Then  $x = \cos \pi = -1$ ,  $y = \sin \pi = 0$  and  $z = 1$ . Note that  $\nabla z = (2x, 2y)$  and  $\nabla z(-1, 0) = (-2, 0)$ . Also  $\vec{r}'(t) = (-\pi \sin \pi t, \pi \cos \pi t, 1)$  and  $\vec{r}'(1) = (0, -\pi, 1)$ . Let  $\theta$  be the angle between  $\vec{r}'(1)$  and the tangent plane to this surface at  $(-1, 0, 1)$ . Then  $\frac{\pi}{2} - \theta$  is the angle between this curve and the normal line to this surface at  $(-1, 0, 1)$ . This normal line is in the direction of  $(-2, 0, -1)$ . Hence

$$\sin \theta = \left| \cos \left( \frac{\pi}{2} - \theta \right) \right| = \frac{|(0, -\pi, 1) \cdot (-2, 0, -1)|}{|(0, -\pi, 1)| |(-2, 0, -1)|} = \frac{|(0)(-2) + (-\pi)(0) + (1)(-1)|}{\sqrt{\pi^2 + 1} \sqrt{5}} = \frac{1}{\sqrt{\pi^2 + 1} \sqrt{5}}$$

**B.** Find the maximum and minimum values of the function  $f(x, y) = \frac{xy}{1+x^2+y^2}$  with domain the disc  $x^2 + y^2 \leq 9$ .

**Solution** First we find all critical points of  $f$  inside the disc.

$$\nabla f = \left( \frac{y(1+x^2+y^2) - xy(2x)}{(1+x^2+y^2)^2}, \frac{x(1+x^2+y^2) - xy(2y)}{(1+x^2+y^2)^2} \right)$$

Hence  $\nabla f = 0$  when  $y(1+x^2+y^2) - 2x^2y = 0$  and  $x(1+x^2+y^2) - 2xy^2 = 0$ .

If  $x = 0$ , then  $y(1+y^2) = 0$  and  $y = 0$ .

If  $y = 0$ , then  $x(1+x^2+y^2) = 0$  and  $x = 0$ .

If  $x$  and  $y$  are both nonzero, then  $1+x^2+y^2 - 2x^2 = 0$  and  $1+x^2+y^2 - 2y^2 = 0$ . Hence  $1 - x^2 + y^2 = 0$  and  $1 + x^2 - y^2 = 0$ . Then  $x^2 - y^2$  is both 1 and  $-1$  which is impossible.

Hence  $(0, 0)$  is the only stationary point.

Parametrize the boundary of the domain by  $r(t) = (3 \cos t, 3 \sin t)$  for  $0 \leq t \leq 2\pi$ . Then

$$F(t) = f(r(t)) = f(3 \cos t, 3 \sin t) = \frac{9 \cos t \sin t}{1 + 9 \cos^2 t + 9 \sin^2 t} = \frac{9}{10} \cos t \sin t = \frac{9}{20} \sin 2t$$

Then  $F'(t) = \frac{9}{10} \cos 2t = 0$  when  $t = \frac{\pi}{4}$ ,  $t = \frac{3\pi}{4}$ ,  $t = \frac{5\pi}{4}$  and  $t = \frac{7\pi}{4}$ . Thus we must consider the points:

$$r\left(\frac{\pi}{4}\right) = \left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right), \quad r\left(\frac{3\pi}{4}\right) = \left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right), \quad r\left(\frac{5\pi}{4}\right) = \left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right) \quad \text{and} \quad r\left(\frac{7\pi}{4}\right) = \left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right).$$

We must compare:

$$f(0, 0) = 0, \quad f\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) = \frac{9/2}{10} = \frac{9}{20}, \quad f\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) = \frac{-9/2}{10} = -\frac{9}{20}, \\ f\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right) = \frac{9/2}{10} = \frac{9}{20}, \quad f\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right) = \frac{-9/2}{10} = -\frac{9}{20}.$$

Thus the minimum value of  $f$  is  $-\frac{9}{20}$  which occurs at  $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$  and at  $\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$ .  
The maximum value of  $f$  is  $\frac{9}{20}$  which occurs at  $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$  and at  $\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$ .