SOLUTIONS TO FINAL EXAM

1. Find the equation of the plane P through the point $(2,-1,4)$ which is perpendicular to the line

$$\frac{x-3}{4} = \frac{y+2}{5} = \frac{7-z}{6} .$$

Solution Rewrite the equation of the line as:

$$\frac{x-3}{4} = \frac{y+2}{5} = \frac{z-7}{-6} .$$

Thus the vector $\overrightarrow{(4,5,-6)}$ is perpendicular to the plane P. The point $(2,-1,4)$ lies on P. Hence P has equation:

$$0 = 4(x-2)+5(y+1)-6(z-4) = 4x+5y-6z+21 .$$

2. Determine whether the following two lines intersect, are parallel or are skew.

$$\left\{\begin{array}{ccc} x & = & 3t-5 \\ y & = & 5-5t \\ z & = & 8t-9 \end{array}\right\} \textit{ and } \left\{\begin{array}{ccc} x & = & 4-3u \\ y & = & 10u-15 \\ z & = & 6-2u \end{array}\right\}$$

Solution We solve the following linear system of three equations in two unknowns:

$$\begin{array}{ccccc} 3t-5 & = & x & = & 4-3u \\ 5-5t & = & y & = & 10u-15 \\ 8t-9 & = & z & = & 6-2u \end{array}$$

Equivalently,

$$\begin{array}{ccc} 3t+3u & = & 9 \\ 5t+10u & = & 20 \\ 8t+2u & = & 15 \end{array}$$

From the first equation: $3t = 9-3u$ and $t = 3-u$. From the second equation: $5(3-u)+10u = 20$. Thus $5u = 5$ and $u = 1$. Then $t = 3-u = 2$. However, $8t+2u$ equals 15 not 18. Hence this linear system of three equations has no solution and these two lines do not intersect. The vector $\overrightarrow{(3,-5,8)}$ is parallel to the first line while the vector $\overrightarrow{(-3,10,-2)}$ is parallel to the second line. Since these vectors are not multiples of each other, they are not parallel and the two lines are not parallel. Hence these lines must be skew.

3. Find parametric equations which parametrize the bottom arc of the hyperbola $\frac{y^2}{4} - \frac{x^2}{9} = 1$ from right to left.

Solution We use the trigonometric identity $\sec^2\theta - \tan^2\theta = 1$ to parametrize this curve.
For $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, let

$$x = \pm 3\tan\theta, \qquad y = \pm 2\sec\theta .$$

Then

$$\frac{y^2}{4} - \frac{x^2}{9} = \frac{(\pm 2\sec\theta)^2}{4} - \frac{(\pm 3\tan\theta)^2}{9} = \sec^2\theta - \tan^2\theta = 1.$$

We take y to be negative so that we get the bottom arc of the hyperbola. Note that $\tan\theta$ is negative for $-\frac{\pi}{2} < \theta < 0$ and positive for $0 < \theta < \frac{\pi}{2}$. Hence we take $x = -\tan\theta$ to move from right to left. Thus the required parametric equations are

$$x = -3\tan\theta, \qquad y = -2\sec\theta$$

for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

4. Find the cosine of the angle of intersection of the curves

$$r(t) = (\cos\pi t,\ \sin\pi t,\ 4t) \ \ and \ \ s(u) = (1-u, 4-u^2, u^3-4)$$

at the point $(-1, 0, 4)$.

Solution Observe that at $(-1, 0, 4)$, $4t = 4$ and $t = 1$. Also $1 - u = -1$ and $u = 2$. Hence we are asked to find the cosine of the angle θ between $r'(1)$ and $s'(2)$. Note that

$$r'(t) = (-\pi\sin\pi t,\ \pi\cos\pi t,\ 4) \ \ and \ \ s'(u) = (-1, -2u, 3u^2)\ .$$

Hence

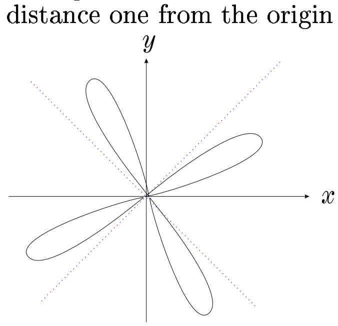
$$r'(1) = (-\pi\sin\pi,\ \pi\cos\pi,\ 4) = (0, -\pi, 4) \ \ and \ \ s'(2) = (-1, -4, 12)\ .$$

Then

$$\cos\theta = \frac{\overrightarrow{(0,-\pi,4)} \cdot \overrightarrow{(-1,-4,12)}}{|\,\overrightarrow{(0,-\pi,4)}\,|\,|\,\overrightarrow{(-1,-4,12)}\,|} = \frac{0 + 4\pi + 48}{\sqrt{0+\pi^2+16}\sqrt{1+16+144}} = \frac{4\pi+48}{\sqrt{161(\pi^2+16)}}$$

5. Plot the polar curve $r^2 = \sin 4\theta$.

Solution Note that this curve only exists for those $0 \le \theta \le 2\pi$ for which $\sin 4\theta \ge 0$, i.e for $0 \le \theta \le \frac{\pi}{4}$, $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$, $\pi \le \theta \le \frac{5\pi}{4}$ and $\frac{3\pi}{2} \le \theta \le \frac{7\pi}{4}$. During each of these intervals there are loops starting and ending at the origin with maximum distance one from the origin at their midpoints.



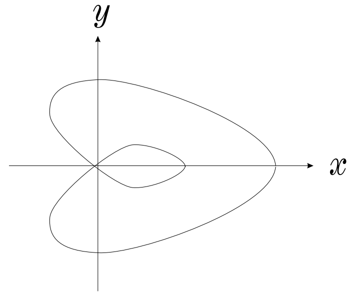
6. Find the area between the inner and outerloops of the limacon $r = 1 + 2\cos\theta$.

Solution Consider the sketch of this limacon below. Note that it passes through the origin when $\cos\theta = -\frac{1}{2}$, i.e for $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. We calculate the area A between the two loops as twice the area inside the upper half of the outside loop minus twice the area inside the upper half of the inside loop.

$$\begin{aligned}
A &= 2\int_0^{\frac{2\pi}{3}} \frac{1}{2}(1+2\cos\theta)^2\, d\theta - 2\int_\pi^{\frac{4\pi}{3}} \frac{1}{2}(1+2\cos\theta)^2\, d\theta \\
&= \int_0^{\frac{2\pi}{3}} 1 + 4\cos\theta + 4\cos^2\theta\, d\theta - \int_\pi^{\frac{4\pi}{3}} 1 + 4\cos\theta + 4\cos^2\theta\, d\theta \\
&= \int_0^{\frac{2\pi}{3}} 1 + 4\cos\theta + 2(1+\cos 2\theta)\, d\theta - \int_\pi^{\frac{4\pi}{3}} 1 + 4\cos\theta + 2(1+\cos 2\theta)\, d\theta \\
&= \int_0^{\frac{2\pi}{3}} 3 + 4\cos\theta + 2\cos 2\theta\, d\theta - \int_\pi^{\frac{4\pi}{3}} 3 + 4\cos\theta + 2\cos 2\theta\, d\theta
\end{aligned}$$

$$= \left(3\theta + 4\sin\theta + \sin 2\theta\Big|_0^{\frac{2\pi}{3}}\right) - \left(3\theta + 4\sin\theta + \sin 2\theta\Big|_{\pi}^{\frac{4\pi}{3}}\right)$$

$$= \left(2\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) - \left(\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}$$



7. Find the length of the curve $r(t) = (e^t,\ e^{-t},\ t\sqrt{2})$ for $0 \le t \le 2$.

Solution By the formula for arclength, the length s of this curve is:

$$\begin{aligned} s &= \int_0^2 \sqrt{[D(e^t)]^2 + [D(e^{-t})]^2 + \left[D(t\sqrt{2})\right]^2}\ dt = \int_0^2 \sqrt{(e^t)^2 + (-e^{-t})^2 + \left(\sqrt{2}\right)^2}\ dt \\ &= \int_0^2 \sqrt{e^{2t} + e^{-2t} + 2}\ dt = \int_0^2 \sqrt{(e^t + e^{-t})^2}\ dt \\ &= \int_0^2 e^t + e^{-t}\ dt = e^t - e^{-t}\Big|_0^2 = (e^2 - e^{-2}) - (1 - 1) = e^2 - e^{-2} \end{aligned}$$

8. Set up a definite integral which gives the circumference of one petal of rose $r = \cos 2\theta$.
DO NOT INTEGRATE.

Solution One petal of this rose occurs for $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$. By the for formula for arclength, the circumference C of one petal is:

$$\begin{aligned} C &= \int_{-\pi/4}^{\pi/4} \sqrt{\cos^2 2\theta + [D(\cos 2\theta)]^2}\ d\theta = \int_{-\pi/4}^{\pi/4} \sqrt{\cos^2 2\theta + (-2\sin 2\theta)^2}\ d\theta \\ &= \int_{-\pi/4}^{\pi/4} \sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\ d\theta = \int_{-\pi/4}^{\pi/4} \sqrt{1 + 3\sin^2 2\theta}\ d\theta \end{aligned}$$

9. Identify each of the following surfaces in xyz–space.
(a) $x^2 - 6x + z = y^2 + 4y$
(b) $x^2 + y^2 = \sin z$
(c) $y = z^2$

Solution **(a)** Complete the square:

$$\begin{aligned} (x-3)^2 - 9 + z &= (y+2)^2 - 4 \\ z - 5 &= (y+2)^2 - (x-3)^2 \end{aligned}$$

Thus this surface is a hyperbolic paraboloid with center $(3, -2, 5)$.
(b) Write this surface as $\left(\sqrt{x^2 + y^2}\right)^2 = \sin z$. Hence this surface is obtained by rotating the curve $x = \sin z$ in the xz–plane around the z–axis. Alternatively, this surface is obtained by rotating the curve $y = \sin z$ in the yz–plane around the z–axis.

(c) Since the variable x is missing from this equation, this surface is the cylinder determined by the parabola $y = z^2$ in the yz–plane.

10. Evaluate

$$\lim_{(x,y)\to(1,0)} \frac{xy - y}{x^2 - 2x + y^2 + 1} .$$

Solution Factor the numerator and complete the square in the denominator:

$$\lim_{(x,y)\to(1,0)} \frac{xy - y}{x^2 - 2x + y^2 + 1} = \lim_{(x,y)\to(1,0)} \frac{y(x-1)}{(x-1)^2 + y^2} .$$

Consider what happens when (x, y) approaches the point $(1, 0)$ along the line $y = m(x - 1)$ for m fixed:

$$\lim_{(x,y)\to(1,0)} \frac{xy - y}{x^2 - 2x + y^2 + 1} = \lim_{x\to 1} \frac{m(x-1)^2}{(x-1)^2 + m^2(x-1)^2} = \lim_{x\to 1} \frac{m}{1+m^2} = \frac{m}{1+m^2} .$$

Since we obtain different limits along different lines, the limit $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy - y}{x^2 - 2x + y^2 + 1}$ does not exist.

11. Find the largest domain and the corresponding range of the function

$$f(x, y, z) = \frac{1}{1 - x^2 - y^2 - z^2} .$$

Solution $f(x, y, z)$ is defined when its denominator is nonzero, i.e. when $1 - x^2 - y^2 - z^2 \neq 0$. Thus the largest domain of f consists of all points which do not lie on the sphere $x^2 + y^2 + z^2 = 1$. The values of the denominator of f are all nonzero numbers less than or equal to one. Their reciprocals are all negative numbers and all positive numbers greater than or equal to one. Thus the range of f is $(-\infty, 0) \cup [1, \infty)$.

12. Find the path of steepest descent on the surface $z = x^2 - y^2$ starting at the point $(3, 2, 5)$.

Solution The projection of this curve $r(t) = (x(t), y(t), z(t))$, $t \geq 0$, into the xy–plane is in the opposite direction of the gradient at each point:

$$\vec{r}'(t) = \overrightarrow{(x'(t), y'(t))} = -K\nabla z = -K\,\overrightarrow{(2x, -2y)} .$$

We can take any positive value for K, say $K = 1$. Then

$$x'(t) = -2x \quad and \quad y'(t) = 2y .$$

Hence $x(t) = x(0)e^{-2t} = 3e^{-2t}$ and $y(t) = y(0)e^{2t} = 2e^{2t}$. The corresponding curve on the surface has $z(t) = x(t)^2 - y(t)^2 = (3e^{-2t})^2 - (2e^{2t})^2 = 9e^{-4t} - 4e^{4t}$. Thus

$$r(t) = \left(3e^{-2t}, 2e^{2t}, 9e^{-4t} - 4e^{4t}\right)$$

for $t \geq 0$.

13. Let $f(x, y) = x^5y^7 - x^6y^4$ with $x = g(s, t) = s^3t^2 + s^5t^4$ and $y = h(s, t) = s^6t^5 - s^8t^7$. Define

$$F(s, t) = f(g(s, t), h(s, t)) .$$

Find ∇F. You may leave your answer in terms of x, y, s and t.

Solution By the chain rule:

$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial s} = (5x^4y^7 - 6x^5y^4)(3s^2t^2 + 5s^4t^4) + (7x^5y^6 - 4x^6y^3)(6s^5t^5 - 8s^7t^7)$$

$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial t} = (5x^4y^7 - 6x^5y^4)(2s^3t + 4s^5t^3) + (7x^5y^6 - 4x^6y^3)(5s^6t^4 - 7s^8t^6)$$

14. Find the equation of the tangent plane to the surface

$$x^3y^2z^4 + x^2y^4z^3 = 12$$

at the point $(2, -1, 1)$.

Solution Let $f(x, y, z) = x^3y^2z^4 + x^2y^4z^3 - 12$. Then $\nabla f(2, -1, 1)$ is perpendicular to the tangent plane P at $(2, -1, 1)$.

$$\begin{aligned} \nabla f &= \overrightarrow{(3x^2y^2z^4 + 2xy^4z^3,\ 2x^3yz^4 + 4x^2y^3z^3,\ 4x^3y^2z^3 + 3x^2y^4z^2)} \\ \nabla f(2, -1, 1) &= \overrightarrow{(16, -32, 44)}\ . \end{aligned}$$

Hence P has equation

$$\begin{aligned} 0 &= 16(x-2) - 32(y+1) + 44(z-1) \\ 108 &= 16x - 32y + 44z \end{aligned}$$

15. Find the minimum and maximum values of $f(x, y) = x^2 + 4xy + y^2 + 12y$ on the region D bounded by the lines $x + y = 2$, $y = x - 2$ and $x = -5$.

Solution First, look for local extrema inside D which are stationary points.

$$f_x = 2x + 4y = 0 \quad and \quad f_y = 4x + 2y + 12 = 0.$$

From the first equation $x = -2y$. Substituting into the second equation: $0 = 4(-2y) + 2y + 12 = 12 - 6y$. Hence $y = 2$ and $x = -4$. Thus $(-4, 2)$ is the only stationary point. Consider the boundary of D in the figure below. We parametrize each side of the triangle.

The segment from $(-5, -7)$ to $(-5, 7)$ is parametrized by $r(t) = (-5, t)$ for $-7 \le t \le 7$. Then

$$A(t) = f(r(t)) = 25 - 20t + t^2 + 12t = 25 - 8t + t^2.$$

$A'(t) = -8 + 2t = 0$ when $t = 4$ at the point $(-5, 4)$.

The segment from $(-5, -7)$ to $(2, 0)$ is parametrized by $s(t) = (t, t - 2)$ for $-5 \le t \le 2$. Then

$$B(t) = f(s(t)) = t^2 + 4t(t-2) + (t-2)^2 + 12(t-2) = t^2 + 4t^2 - 8t + t^2 - 4t + 4 + 12t - 24 = 6t^2 - 20.$$

$B'(t) = 12t = 0$ when $t = 0$ at the point $(0, -2)$.

The segment form $(-5, 7)$ to $(2, 0)$ is parametrized by $u(t) = (t, 2 - t)$ for $-5 \le t \le 2$. Then

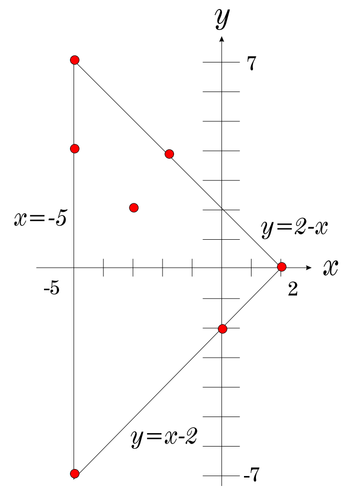
$$C(t) = f(u(t)) = t^2 + 4t(2-t) + (2-t)^2 + 12(2-t) = t^2 + 8t - 4t^2 + 4 - 4t + t^2 + 24 - 12t = -2t^2 - 8t + 28.$$

$C'(t) = -4t - 8 = 0$ when $t = -2$ at the point $(-2, 4)$.

We compare the values at the above four points and at the three corners of the triangle:

$$\begin{array}{lll} f(-4,2) = 16 - 32 + 4 + 24 = 12 & f(-5,4) = 25 - 80 + 16 + 48 = 9 & f(0,-2) = 0 + 0 + 4 - 24 = -20 \\ f(-2,4) = 4 - 32 + 16 + 48 = 36 & f(-5,-7) = 25 + 140 + 49 - 84 = 130 & f(-5,7) = 25 - 140 + 49 + 84 = 18 \\ f(2,0) = 4 + 0 + 0 + 0 = 4 & & \end{array}$$

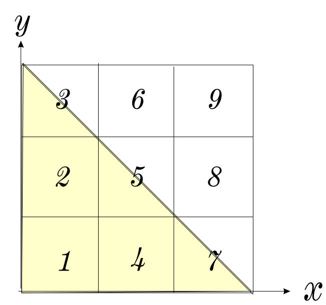
Thus f has its minimum value of -20 at $(0, -2)$ and its maximum value of 130 at $(-5, -7)$.



16. Let B be the region bounded by the line $x+y=3$, the x–axis and the y–axis. Note that B is contained in the square $R=[0,3]\times[0,3]$. Let $P=Q$ be the regular partition of $[0,3]$ into three subintervals. Let $f(x,y)=x+y$. Find the lower Riemann sum $L(P\times Q,f)$ and the upper Riemann sum $U(P\times Q,f)$ for $\iint_B x+y\;dx\;dy$.

Solution This partition $P\times Q$ subdivides the square R into nine squares of area one as in the figure below. We compute for these squares as ordered in the figure:

$$\begin{aligned} L(P\times Q,f) &= f(0,0)(1)+f(0,1)(1)+f(1,2)(2)+f(1,0)(1)+f(2,2)(1)\\ &\quad +f(2,3)(1)+f(3,1)(1)+f(3,2)(1)+f(3,3)(1)\\ &= 0+1+0+1+0+0+0+0+0=2\\ U(P\times Q,f) &= f(1,1)(1)+f(1,2)(1)+f(1,2)(2)+f(2,1)(1)+f(2,1)(1)\\ &\quad +f(1,2)(1)+f(2,1)(1)+f(2,1)(1)+f(3,2)(1)\\ &= 2+3+3+3+3+3+3+3+0=23 \end{aligned}$$



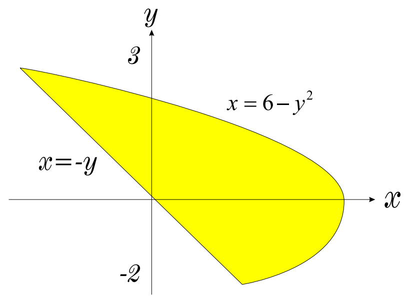
17. Let B be the region bounded by $y=-x$ and $x=6-y^2$. Evaluate $\iint_B xy\;dx\;dy$.
DO NOT SIMPLIFY THE ARITHMETIC IN YOUR ANSWER.

Solution Observe that $y=-x$ and $x=6-y^2$ intersect when $-y=6-y^2$, i.e when $0=y^2-y-6=x=(y-3)(y+2)$ and $y=3$, $y=-2$. Apply Fubini's Theorem integrating first with respect to y. See the figure below.

$$\begin{aligned} \iint_B xy\;dx\;dy &= \int_{-2}^{3}\left[\int_{-y}^{6-y^2} xy\;dx\right]dy \;=\; \int_{-2}^{3}\left[\frac{1}{2}x^2y\Big|_{x=-y}^{x=6-y^2}\right]dy\\ &= \frac{1}{2}\int_{-2}^{3}(6-y^2)^2y-y^3\;dy \;=\; \frac{1}{2}\int_{-2}^{3}36y-13y^3+y^5\;dy \end{aligned}$$

$$= \frac{1}{2}\left(18y^2 - \frac{13}{4}y^4 + \frac{1}{6}y^6\Big|_{-2}^{3}\right)$$

$$= \frac{1}{2}\left[18(9) - \frac{13}{4}(81) + \frac{1}{6}(729)\right] - \frac{1}{2}\left[18(4) - \frac{13}{4}(16) + \frac{1}{6}(64)\right]$$

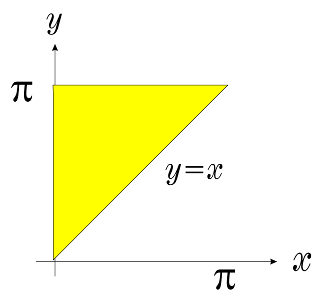


18. Evaluate $\int_0^\pi \left[\int_x^\pi \frac{\sin y}{y}\, dy\right] dx$.

Solution It is impossible to integral $\frac{\sin y}{y}$, so we reverse the order of integration. From the figure below, the region of integration starts from $x = 0$ on the left to $x = y$ on the right. It runs from $y = 0$ to $y = \pi$ on the y–axis. Therefore

$$\int_0^\pi \left[\int_x^\pi \frac{\sin y}{y}\, dy\right] dx = \int_0^\pi \left[\int_0^y \frac{\sin y}{y}\, dx\right] dy = \int_0^\pi \frac{\sin y}{y} x\Big|_0^y dy = \int_0^\pi \frac{\sin y}{y} y\, dy$$

$$= \int_0^\pi \sin y\, dy = -\cos y|_0^\pi = -\cos\pi + \cos 0 = -(-1) + 1 = 2$$



19. Set up a triple integral which gives the volume V of the solid G bounded by the paraboloid $x = y^2 + z^2$ and the paraboloid $6 - x = y^2 + z^2$.
DO NOT INTEGRATE.

Solution These two paraboloids intersect when $y^2 + z^2 = x = 6 - y^2 - z^2$, i.e. when $2y^2 + 2z^2 = 6$ or $y^2 + z^2 = 3$. Hence we integrate over this circle in the yz–plane with the first paraboloid at the back and the second paraboloid at the front.

$$V = \int_{y=-\sqrt{3}}^{y=\sqrt{3}} \left[\int_{z=-\sqrt{3-y^2}}^{z=\sqrt{3-y^2}} \left[\int_{x=y^2+z^2}^{x=6-y^2-z^2} 1\, dx\right] dz\right] dy$$

20. Change the order of ingtegration to rewrite the following sum of iterated integrals as one interated integral.

$$S = -\int_{z=0}^{z=1}\left[\int_{y=1-z}^{y=z-1}\left[\int_{x=-\sqrt{(z-1)^2-y^2}}^{x=\sqrt{(z-1)^2-y^2}} x^2y^2z^2\ dx\right]dy\right]dz$$
$$+\int_{z=0}^{z=1}\left[\int_{y=-\sqrt{1-z^2}}^{y=\sqrt{1-z^2}}\left[\int_{x=-\sqrt{1-y^2-z^2}}^{x=\sqrt{1-y^2-z^2}} x^2y^2z^2\ dx\right]dy\right]dz$$

DO NOT INTEGRATE.

Solution The first integral refers to the the surface $x^2 = (z-1)^2 - y^2$ which is the cone $(z-1)^2 = x^2 + y^2$. The integral describes the entire bottom nappe of this cone from the xy–plane to its vertex at $z = 1$. The second integral refers to the surface $x^2 = 1 - y^2 - z^2$ which is the sphere $x^2 + y^2 + z^2 = 1$. That integral describes the entire upper hemisphere from $z = 0$ to $z = 1$. Thus the solid over which we are integrating is the hemisphere with the cone removed. We can describe it as one integral by integrating first in the z direction from the bottom nappe of the cone on the bottom to the hemisphere on top. These surfaces intersect on the xy–plane $z = 0$ in the circle $x^2 + y^2 = 1$. Thus our second integral will be in the y direction from the lower semi–circle $y = -\sqrt{1-x^2}$ to the upper semi–circle $y = \sqrt{1-x^2}$.

$$S = \int_{x=-1}^{x=1}\left[\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\left[\int_{z=1-\sqrt{x^2+y^2}}^{z=\sqrt{1-x^2-y^2}} x^2y^2z^2\ dz\right]dy\right]dx$$