SOLUTIONS TO EXAM 2

Answer the first question and seven of the other nine questions.
The first question is worth 9 marks and the other questions are worth 13 marks each.
Show all of your work.
Justify each of your answers.

1. State the Second Vector Form of Green's Theorem.

Solution Let D be a simply connected region in $\Re^2$ with boundary the Jordan curve C. Let $\vec{F}$ be a C^1–vector field with domain an open region containing D. Then

$$\int_C \vec{F} = \iint_D \vec{F} \cdot \mathbf{k} \; dx \; dy \; .$$

2. Find a definite integral which gives the value of the line integral

$$\int_C x^2 y^2 \; dx - (x+y) \; dy$$

where C is the clockwise semi–circle of radius two and center the origin which starts at $(-2, 0)$ and ends at $(2, 0)$.
DO NOT INTEGRATE.

Solution Parametrize C by:

$$r(t) = (-2\cos t, \; 2\sin t)$$

for $0 \le t \le \pi$. Then $dx \;=\; 2\sin t \; dt$ and $dy \;=\; 2\cos t \; dt$. Thus

$$\begin{aligned}\int_C x^2 y^2 \; dx - (x+y) \; dy \;&=\; \int_0^{\pi} (-2\cos t)^2 (2\sin t)^2 (2\sin t) - (-2\cos t + 2\sin t)(2\cos t) \; dt \\ &=\; \int_0^{\pi} 32\cos^2 t \sin^3 t + 4\cos^2 t - 4\sin t\cos t \; dt \; .\end{aligned}$$

3. Find a definite integral which gives the value of the line integral

$$\int_C xyz$$

where C is the line segment from the point $(2, -1, 4)$ to the point $(3, 1, -5)$.
DO NOT INTEGRATE.

Solution The vector from the point $(2, -1, 4)$ to the point $(3, 1, -5)$ is $\overrightarrow{(3-1, 1-(-1), -5-4)} = \overrightarrow{(2, 2, -9)}$.
Thus parametrize C by:

$$r(t) = (2+2t, \; -1+2t, \; 4-9t)$$

for $0 \le t \le 1$. Then $\vec{r}\,'(t) = (2, 2, -9)$ and

$$\begin{aligned}\int_C xyz \;&=\; \int_0^1 (1+2t)(2t-1)(4-9t)\sqrt{2^2+2^2+(-9)^2} \; dt \;=\; \int_0^1 (4t^2-1)(4-9t)\sqrt{89} \; dt \\ &=\; \sqrt{89}\int_0^1 -36t^3 + 16t^2 + 9t - 4 \; dt \; .\end{aligned}$$

4. Let $P = (-1, 0)$, $Q = (0, 1)$, $R = (1, 0)$ with C the boundary $PQ\sharp QR\sharp RP$ of the triangle PQR. Find a double integral which gives the value of the line integral

$$\int_C x^3y^2\ dx - x^2y^3\ dy\ .$$

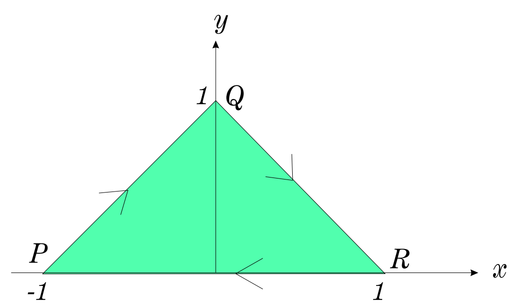
DO NOT INTEGRATE.

Solution Apply Green's Theorem to the triangle $T = PQR$. Since the boundary C of this triangle has a clockwise orientation:

$$\begin{aligned}\int_C x^3y^2\ dx - x^2y^3\ dy &= -\iint_T \frac{\partial}{\partial x}(-x^2y^3) - \frac{\partial}{\partial y}(x^3y^2)\ dx\ dy\ =\ -\iint_T -2xy^3 - 2x^3y\ dx\ dy \\ &= \iint_T 2xy^3 + 2x^3y\ dx\ dy\ .\end{aligned}$$

Note from the figure below that T can be described from left to right as going from the line $x = y - 1$ to the line $x = 1 - y$ for $0 \le y \le 1$. Hence

$$\int_C x^3y^2\ dx - x^2y^3\ dy\ =\ \int_{y=0}^{y=1}\int_{x=y-1}^{x=1-y} 2xy^3 + 2x^3y\ dx\ dy\ .$$



5. Let C be the polygonal path $P_1P_2 \cup P_2P_3 \cup P_3P_4 \cup P_4P_5$ with

$$P_1 = (2, 1, -3),\quad P_2 = (4, -2, 0),\quad P_3 = (-1, 8, 7),\quad P_4 = (3, -5, 6),\quad P_5 = (1, -1, 2).$$

Evaluate $\int_C xy^2z^2\ dx + x^2yz^2\ dy + x^2y^2z\ dz$.

Solution Observe that the vector field $\overrightarrow{(xy^2z^2, x^2yz^2, x^2y^2z)}$ is the gradient of the scalar field $f(x, y, z) = \frac{1}{2}x^2y^2z^2$. Hence

$$\begin{aligned}\int_C xy^2z^2\ dx + x^2yz^2\ dy + x^2y^2z\ dz &= f(P_5) - f(P_1)\ =\ f(1, -1, 2) - f(2, 1, -3) \\ &= \frac{1}{2}(1)^2(-1)^2(2)^2 - \frac{1}{2}(2)^2(1)^2(-3)^2\ =\ 2 - 18\ =\ -16\ .\end{aligned}$$

6. Determine whether the following vector field

$$\vec{F}\ = \overrightarrow{(4y^3z^2 - 8xyz^3,\ 12xy^2z^2 - 4x^2z^3,\ 8xy^3z - 12x^2yz^2)}$$

is solenoidal or irrotational.

Solution Observe that

$$\begin{aligned}\text{div}\vec{F} &= \frac{\partial}{\partial x}(4y^3z^2 - 8xyz^3) + \frac{\partial}{\partial y}(12xy^2z^2 - 4x^2z^3) + \frac{\partial}{\partial z}(8xy^3z^3 - 12x^2yz^2) \\ &= 0 - 8yz^3 + 24xyz^2 - 0 + 8xy^3 - 24x^2y^2z \;=\; -8yz^3 + 24xyz^2 + 8xy^3 - 24x^2y^2z \;.\end{aligned}$$

Since div$\vec{F}$ is not zero, the vector field $\vec{F}$ is not solenoidal. Note that

$$\begin{aligned}\text{curl}\vec{F} &= \left[\frac{\partial}{\partial y}(8xy^3z - 12x^2yz^2) - \frac{\partial}{\partial z}(12xy^2z^2 - 4x^2z^3)\right]\mathbf{i} \\ &\quad + \left[\frac{\partial}{\partial z}(4y^3z^2 - 8xyz^3) - \frac{\partial}{\partial x}(8xy^3z - 12x^2yz^2)\right]\mathbf{j} \\ &\quad + \left[\frac{\partial}{\partial x}(12xy^2z^2 - 4x^2z^3) - \frac{\partial}{\partial y}(4y^3z^2 - 8xyz^3)\right]\mathbf{k} \\ &= \left[(24xy^2z - 12x^2z^2) - (24xy^2z - 12x^2z^2)\right]\mathbf{i} + \left[(8y^3z - 24xyz^2) - (8y^3z - 24xyz^2)\right]\mathbf{j} \\ &\quad + \left[(12y^2z^2 - 8xz^3) - (12y^2z^2 - 8xz^3)\right]\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} \;=\; 0 \;.\end{aligned}$$

Since curl$\vec{F} = 0$, the vector field $\vec{F}$ is irrotational.

7. Let S be the back cap of the sphere $x^2 + y^2 + z^2 = 4$ for $-2 \le x \le -1$ with outward orientation. Find parametric equations which describe S as a basic parametric surface.

Solution The back hemisphere of this sphere is the graph of $x = f(y,z) = -\sqrt{4 - y^2 - z^2}$ with domain the disc $y^2 + z^2 \le 2$ in the yz–plane. The back cap is bounded by

$$\begin{aligned} -1 &= x = -\sqrt{4 - y^2 - z^2} \\ 1 &= 4 - y^2 - z^2 \\ y^2 + z^2 &= 3 \;.\end{aligned}$$

That is, the back cap S is the graph of the restriction of f to the disc $y^2 + z^2 \le 3$. Consider the basic parametric surface

$$s(t,u) \;=\; (-\sqrt{4 - t^2 - u^2},\; t,\; u)$$

with domain the disc D of radius $\sqrt{3}$ and center the origin. The orientation induced by s is given by

$$\vec{N} \;=\; (1,\; -\vec{f_t},\; -f_u) \;=\; \left(1,\; \frac{t}{\sqrt{4 - t^2 - u^2}},\; \overrightarrow{\frac{u}{\sqrt{4 - t^2 - u^2}}}\right)$$

which points frontwards because the first coordinate is positive. This is the inward orientation. Hence give S the parametrization

$$r(t,u) \;=\; (-\sqrt{4 - t^2 - u^2},\; u,\; t)$$

for $(t,u) \in D$. Let $P = -\sqrt{4-t^2-u^2}$, $Q = u$ and $R = t$. The orientation induced by r is

$$
\begin{aligned}
\vec{N} &= \left(\frac{\partial(Q,R)}{\partial(t,u)}, \frac{\partial(R,P)}{\partial(t,u)}, \frac{\partial(P,Q)}{\partial(t,u)} \right) \\
&= \left(\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ \frac{t}{\sqrt{4-t^2-u^2}} & \frac{u}{\sqrt{4-t^2-u^2}} \end{vmatrix}, \begin{vmatrix} \frac{t}{\sqrt{4-t^2-u^2}} & \frac{u}{\sqrt{4-t^2-u^2}} \\ 0 & 1 \end{vmatrix} \right) \\
&= \left(-1, \frac{u}{\sqrt{4-t^2-u^2}}, \frac{t}{\sqrt{4-t^2-u^2}} \right)
\end{aligned}
$$

which points backwards because the first coordinate is negative. This is the outward orientation. Therefore r is the required parametrization of S.

8. Consider the surface $S = S_1 \cup S_2$.
S_1 has parametrization

$$s_1(t,u) = (t,u,1)$$

for (t,u) in the square $[0,1] \times [0,1]$. S_2 has parametrization

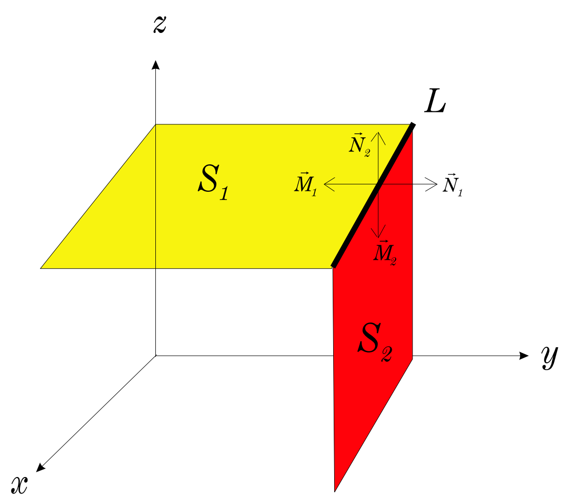
$$s_2(t,u) = (u,1,t)$$

for (t,u) in the square $[0,1] \times [0,1]$. Determine whether S_1 and S_2 have compatible orientations on $S_1 \cap S_2$.

Solution Consider the sketch of S below. S_1 is the graph of the function $z = 1$ with domain $[0,1] \times [0,1]$ in the xy–plane. Hence the orientation of S_1 is $\vec{N}_1 = (0,0,1)$. That is, the horizontal square S_1 has orientation $\vec{N}_1 = \mathbf{k}$.
S_2 is the graph of the function $y = 1$ with domain $[0,1] \times [0,1]$ in the xz–plane. Hence the orientation of S_2 is $\vec{N}_2 = (0,1,0)$. That is, the vertical square S_2 has orientation $\vec{N}_2 = \mathbf{j}$.
Note that $L = S_1 \cap S_2$ is the line segment of points $(x,1,1)$ for $0 \le x \le 1$.
Note that $\vec{M}_1$ is the horizontal unit vector which points inwards into S_1 on each point of L. Thus $\vec{M}_1$ points in the opposite direction of the y–axis and $\vec{M}_1 = -\mathbf{j}$. The vertical vector $\vec{M}_2$ points inwards into S_2 on each point of L. That is, $\vec{M}_2$ points downwards, i.e. $\vec{M}_2 = -\mathbf{k}$. Hence

$$
\begin{aligned}
\vec{M}_1 \times \vec{N}_1 &= -\mathbf{j} \times \mathbf{k} = -\mathbf{i} \\
\vec{M}_2 \times \vec{N}_2 &= -\mathbf{k} \times \mathbf{j} = \mathbf{i}\ .
\end{aligned}
$$

Since these are opposite orientations of L, it follows that S_1 and S_2 have compatible orientations.



9. Let S be the oriented surface parametrized by

$$s(t,u) \;=\; (t+u,\; u-t,\; tu)\ .$$

for $(t,u) \in [0,3] \times [0,2]$. Let $\vec{F}$ be the vector field

$$\vec{F} \;=\; \overrightarrow{(xy,\; xz,\; yz)}\ .$$

Find a double integral whose value is the surface integral

$$\int_S \vec{F} \cdot \vec{N}\ .$$

DO NOT SIMPLIFY THE INTEGRAND AND DO NOT INTEGRATE.

Solution Let $P = t+u$, $Q = u-t$ and $R = tu$. Then

$$\begin{aligned}
\iint_S \vec{F}\cdot\vec{N} &= \int_0^3\int_0^2 xy\frac{\partial(Q,R)}{\partial(t,u)} + xz\frac{\partial(R,P)}{\partial(t,u)} + yz\frac{\partial(P,Q)}{\partial(t,u)}\; du\; dt \\
&= \int_0^3\int_0^2 (t+u)(u-t)\begin{vmatrix} -1 & 1 \\ u & t \end{vmatrix} + (t+u)(tu)\begin{vmatrix} u & t \\ 1 & 1 \end{vmatrix} + (u-t)(tu)\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}\; du\; dt \\
&= \int_0^3\int_0^2 (t+u)(u-t)(-t-u) + (t+u)(tu)(u-t) + (u-t)(tu)(2)\; du\; dt
\end{aligned}$$

10. Find a double integral in polar coordinates whose value is the surface area A of the portion S of the hyperboloid $z^2 - x^2 - y^2 = 1$ for $1 \le z \le 3$.

Solution This hyperboloid is defined by the implicit equation $g(x,y,z) = z^2 - x^2 - y^2 - 1 = 0$ for $1 \le z \le 3$, i.e. for

$$\begin{aligned}
1 &\le \sqrt{1+x^2+y^2} \le 3 \\
1 &\le 1+x^2+y^2 \le 9 \\
0 &\le x^2+y^2 \le 8
\end{aligned}$$

which is the disc D in the xy–plane with center the origin and radius $\sqrt{8} = 2\sqrt{2}$. The surface area of S is given by:

$$\begin{aligned} A &= \iint_D \frac{\sqrt{g_x^2 + g_y^2 + g_z^2}}{|g_z|}\, dx\, dy = \iint_D \frac{\sqrt{(-2x)^2 + (-2y)^2 + (2z)^2}}{2z}\, dx\, dy \\ &= \iint_D \frac{\sqrt{4x^2 + 4y^2 + 4z^2}}{2z}\, dx\, dy = \iint_D \frac{\sqrt{4x^2 + 4y^2 + 4(1 + x^2 + y^2)}}{2\sqrt{1 + x^2 + y^2}}\, dx\, dy \\ &= \iint_D \frac{1}{2}\sqrt{\frac{4 + 8x^2 + 8y^2}{1 + x^2 + y^2}}\, dx\, dy = \iint_D \sqrt{\frac{1 + 2x^2 + 2y^2}{1 + x^2 + y^2}}\, dx\, dy\ . \end{aligned}$$

We could put in the bounds of this integral using xy–coordinates, but the description is simpler in polar coordinates. The circle D is described by letting r vary from 0 to $2\sqrt{2}$ and letting θ vary from 0 to 2π. Recall that the Jacobian for polar coordinates is r. Hence

$$A = \int_0^{2\pi} \int_0^{2\sqrt{2}} \sqrt{\frac{1 + 2r^2}{1 + r^2}}\, r\, dr\, d\theta\ .$$