SOLUTIONS TO EXAM 2

1. Use spherical coordinates to find the volume V of the cored apple which consists of a sphere of radius two with a cylindrical hole of radius one. (The axis of the cylinder passes through the center of the sphere.)

Solution The apple has spherical equation $\rho = 2$ while the core has equation

$$\begin{aligned} 1 &= x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) = \rho^2 \sin^2\phi \\ 1 &= \rho \sin\phi \\ \rho &= \csc\phi \ . \end{aligned}$$

Thus the lower bound of ρ is the cylinder $\rho = \csc\phi$ and the upper bound of ρ is the sphere $\rho = 2$. The cylinder and the sphere intersect when

$$\begin{aligned} \csc\phi &= \rho \ = \ 2 \\ \sin\phi &= \frac{1}{2} \\ \phi &= \frac{\pi}{6}, \frac{5\pi}{6} \ . \end{aligned}$$

Hence the lower bound of ϕ is $\phi = \frac{\pi}{6}$ while the upper bound of ϕ is $\phi = \frac{5\pi}{6}$. The values of θ are its entire range $0 \le \theta \le 2\pi$. Thus

$$\begin{aligned} V &= \int_0^{2\pi} \left[\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[\int_{\csc\phi}^{2} 1 \cdot \rho^2 \sin\phi \, d\rho \right] d\phi \right] d\theta \ = \ \int_0^{2\pi} \left[\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{3}\rho^3 \sin\phi \Big|_{\csc\phi}^{2} \, d\phi \right] d\theta \\ &= \int_0^{2\pi} \left[\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{8}{3} \sin\phi - \frac{1}{3} \csc^3\phi \sin\phi \, d\phi \right] d\theta \ = \ \int_0^{2\pi} \left[\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{8}{3} \sin\phi - \frac{1}{3} \csc^2\phi \, d\phi \right] d\theta \\ &= \int_0^{2\pi} -\frac{8}{3} \cos\phi + \frac{1}{3} \cot\phi \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \, d\theta \\ &= \int_0^{2\pi} -\frac{8}{3} \cos\frac{5\pi}{6} + \frac{1}{3} \cot\frac{5\pi}{6} + \frac{8}{3} \cos\frac{\pi}{6} - \frac{1}{3} \cot\frac{\pi}{6} \, d\theta \\ &= \int_0^{2\pi} \frac{8}{3}\frac{\sqrt{3}}{2} - \frac{1}{3}\sqrt{3} + \frac{8}{3}\frac{\sqrt{3}}{2} - \frac{1}{3}\sqrt{3} \, d\theta \ = \ \int_0^{2\pi} 2\sqrt{3} \, d\theta \ = \ 2\sqrt{3}(2\pi) \ = \ 4\pi\sqrt{3} \ . \end{aligned}$$

2. Consider the curve $\vec{r}(u) = \overrightarrow{(u^2, 2u - 1)}$, for $u \in [0, 6]$.
Let $P = \{0, 2, 4, 6\}$ be a partition of $[0, 6]$, let $T = \{1, 3, 5\}$ and let $f(x, y) = xy$.
Evaluate the Riemann-Stieltjes sum $RS(P, T, f, \vec{r})$.

Solution $RS(P, T, f, \vec{r})$ has three summands constructed using $u_0 = 0$, $u_1 = 2$, $u_2 = 4$, $u_3 = 6$ and $t_1 = 1$, $t_2 = 3$, $t_3 = 5$:

$$RS(P, T, f, \vec{r}) \ = \ f(r(t_1)) | \overrightarrow{r(u_0)r(u_1)} | + f(r(t_2)) | \overrightarrow{r(u_1)r(u_2)} | + f(r(t_3)) | \overrightarrow{r(u_2)r(u_3)} |$$

$$
\begin{aligned}
&= f(r(1))|\overrightarrow{r(0)r(2)}| + f(r(3))|\overrightarrow{r(2)r(4)}| + f(r(5))|\overrightarrow{r(4)r(6)}| \\
&= f(1,1))|\overrightarrow{(0,-1)(4,3)}| + f(9,5)|\overrightarrow{(4,3)(16,7)}| + f(25,9)|\overrightarrow{(16,7)(36,11)}| \\
&= |\overrightarrow{(4,4)}| + 45|\overrightarrow{(12,4)}| + 225|\overrightarrow{(20,4)}| \;=\; 4|\overrightarrow{(1,1)}| + 45(4)|\overrightarrow{(3,1)}| + 225(4)|\overrightarrow{(5,1)}| \\
&= 4\sqrt{2} + 180\sqrt{10} + 900\sqrt{26}
\end{aligned}
$$

3. Evaluate the line integral

$$I \;=\; \int_C (x+y+z)\ dx + (x-y-z)\ dy + (y-x+z)\ dz$$

where C is the line segment from $(3,-4,6)$ to $(5,1,2)$.

Solution Parametrize C by $r(t) = (3+2t, -4+5t, 6-4t)$ for $0 \le t \le 1$.
Then $r'(t) = (2,5,-4)$. Hence

$$
\begin{aligned}
I \;&=\; \int_0^1 [(3+2t)+(-4+5t)+(6-4t)]\,(2) + [(3+2t)-(-4+5t)-(6-4t)]\,(5) \\
&\qquad\qquad + [(-4+5t)-(3+2t)+(6-4t)]\,(-4)\ dt \\
&= \int_0^1 (3t+5)(2)+(t+1)(5)+(-t-1)(-4)\ dt \;=\; \int_0^1 6t+10+5t+5+4t+4\ dt \\
&= \int_0^1 15t+19\ dt \;=\; \frac{15}{2}t^2+19t\Big|_0^1 \;=\; \frac{15}{2}+19 \;=\; \frac{53}{2}\ .
\end{aligned}
$$

4. Evaluate the line integral

$$J = \int_C 5xy\ dx + 7x^2y\ dy$$

where C is the clockwise boundary of the square $[-1,2]\times[0,3]$.

Solution Apply Green's Theorem to the square $S = [-1,2]\times[0,3]$ with its counterclockwise boundary $-C$:

$$
\begin{aligned}
J \;&=\; -\int_{-C} 5xy\ dx + 7x^2y\ dy \;=\; \iint_S \frac{\partial}{\partial x}(7x^2y) - \frac{\partial}{\partial y}(5xy)\ dx\ dy \\
&= \int_{-1}^{2}\left[\int_0^3 14xy-5x\ dy\right] dx \;=\; \int_{-1}^{2} 7xy^2-5xy\Big|_0^3\ dx \;=\; \int_{-1}^{2} 63x-15x\ dx \\
&= \int_{-1}^{2} 48x\ dx \;=\; 24x^2\Big|_{-1}^{2} \;=\; (24)(16)-24(1) \;=\; 360
\end{aligned}
$$

5. Answer each of these short questions.
(a) Let $u = x+y$ and $v = x-y$ be a change of coordinates in the xy–plane. Let the graph of $z = f(u,v)$ be a surface S which lies above the region R in the xy–plane bounded by

the curves $u = 1$, $u = 3$, $v = 2$, $v = 6$. Write an iterated integral which gives the volume V of the solid which is bounded above by S, bounded below by R and whose sides are the cylinder through the boundary of R.

Solution We must solve for x and y in terms of u and v:

$$\begin{aligned} u+v &= (x+y)+(x-y) = 2x \\ u-v &= (x+y)-(x-y) = 2y \end{aligned}$$

Hence

$$x = \frac{1}{2}(u+v), \qquad y = \frac{1}{2}(u-v) .$$

Then

$$\begin{aligned} V &= \int_1^3 \left[\int_2^6 f(u,v) \left| \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \right| dv \right] du = \int_1^3 \left[\int_2^6 f(u,v) \left| \det \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \right| dv \right] du \\ &= \int_1^3 \left[\int_2^6 f(u,v) \left| -\frac{1}{2} \right| dv \right] du = \int_1^3 \left[\int_2^6 \frac{1}{2} f(u,v)\, dv \right] du . \end{aligned}$$

(b) Evaluate $I = \int_C 1 \, d\vec{r}$ where $\vec{r}(t) = \overrightarrow{(3\cos(\pi t^4),\, 3\sin(\pi t^4))}$ for $0 \le t \le 1$.

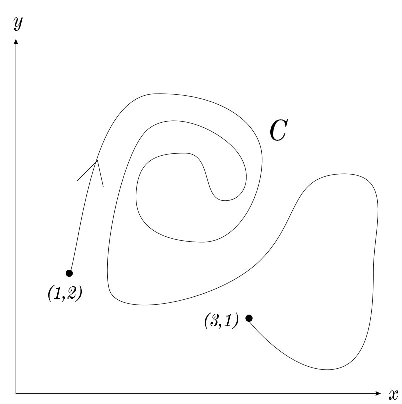
Solution Note that C is the upper semi–circle of $x^2 + y^2 = 9$. Then I is the length of C which is 3π.

(c) Let C_k denote the boundary of the square $[-k, k] \times [-k, k]$ with a counterclockwise orientation. Evaluate

$$I = \int_{C_4} x \, dy + \int_{C_2} y \, dx .$$

Solution Observe that $\int_{C_4} x \, dy$ is the area of C_4, and $-\int_{C_2} y \, dx = \int_{C_2} -y \, dx$ is the area of C_2. Hence $I = 64^2 - 4^2 = 64 - 16 = 48$.

(d) Evaluate the line integral $J = \int_C y \, dx + x \, dy$ where C is depicted in the following diagram.



Solution Observe that $\overrightarrow{(y,x)} = \nabla f$ where $f(x,y) = xy$. Hence the value of the line integral, J, is independent of the path from $(1,2)$ to $(3,1)$, and has value

$$J = f(3,1) - f(1,2) = (3)(1) - (1)(2) = 1.$$