DIFFERENTIAL OPERATORS AND WEIGHTED ISOBARIC POLYNOMIALS

TRUEMAN MACHENRY AND GEANINA TUDOSE

Abstract. We characterize those sequences of weighted isobaric polynomials [5] which belong to the kernel of the linear operator $D_{11} - \sum_{j=1}^{k} a_j t_j D_{2j} - mD_2, \geq 2$, and we characterize those linear operators of this form in terms of the coefficients a_j which have a non-zero kernel.

1. Introduction

In [4] the following linear operator was introduced:

$$\mathfrak{T}_m = D_{11} - \sum_j t_j D_{2j} - mD_2,$$

where $m \in \mathbb{Z}$. We are interested in these operators as linear operators on a special ring of polynomials, discussed in [5], namely, the ring of *isobaric polynomials* $\tilde{\Lambda}$, a ring isomorphic to the ring of symmetric functions Λ. The polynomials in $\tilde{\Lambda}$, or more precisely $\tilde{\Lambda}_k$, are over indeterminants $t_1, \dots, t_k$ and the isomorphism just mentioned is given by identifying t_j with the signed elementary symmetric polynomial $(-1)^{j+1}e_j$. This determines an involutory mapping whose elements in $\tilde{\Lambda}$ are called isobaric reflects.

Isobaric polynomials can be defined independently of Λ as follows: for each n, let $P_{k,n} := \sum_\alpha A_\alpha t^\alpha$ where $\alpha = (\alpha_1, \dots, \alpha_k)$, $t^\alpha = t_1^{\alpha_1} \dots t_k^{\alpha_k}$, and $(1^{\alpha_1}, \dots, k^{\alpha_k})$ is a partition of n, n remaining constant for all monomials in the polynomial. We call this a polynomial of *isobaric degree n* (even though the coefficient of the term of maximal degree n may be zero). In general the A_α can be taken from any commutative ring. In what follows we restrict ourselves to the ring of integers. It is our purpose in this paper to discuss the portion of the kernel of $\mathfrak{T}_m$ lying in $\tilde{\Lambda}$, or rather, more precisely, we are interested in certain sequences of polynomials in $\tilde{\Lambda}$ all of whose entries lie in the kernel of $\mathfrak{T}_m$. These are the sequences of polynomials determined by a certain weighting operation whose entries lie in the module of Weighted Isobaric Polynomials (WIP-module) in $\tilde{\Lambda}$ as defined in [5].

In [4] it was shown that two such sequences are the sequence of Generalized Fibonacci Polynomials $\{F_n\}_n$, for $m = 2$, and the sequence of Generalized Lucas Polynomials $\{G_n\}_n$, for $m = 1$, which are, respectively, reflects of the complete symmetric polynomials (CSP) and of the power symmetric polynomials (PSP)(see also [1],[2],[3]. In this paper we shall answer the following questions.

(1) Are there any other WIP sequences of solutions for these two values of m ?

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(2) Are there any other WIP sequences of solutions for other integer values of m ?

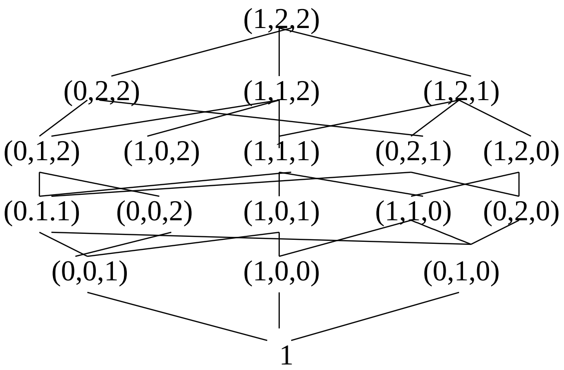
(3) Are there any reasonable generalizations of these partial differential equations for which there are WIP sequences of solutions ?

1.1. **The Differential Lattice.** With each monomial $t_1^{\alpha_1}\dots t_k^{\alpha_k}$ we associate a lattice $\mathcal{L}(t)$ as follows: the top of the lattice is the node $t_1^{\alpha_1}\dots t_k^{\alpha_k}$, an element of depth $\sum_i \alpha_i$. The elements of depth $\sum_i \alpha_i - j$ are those in which some monomial $t^\beta = t_1^{\beta_1}\dots t_k^{\beta_k}$ of depth $\sum_i(\alpha_i - j + 1)$ has been replaced by a monomial $t_1^{\beta_1}\dots t_i^{\beta_i - 1}\dots t_k^{\beta_k}$. Depth 1 consists of nodes of monomials $t_1, \dots, t_k$. Two nodes are connected by an edge if one node is derived from the other by subtracting 1 from the exponent sum. One node is less then another node if its depth is smaller and if the two nodes are connected by a sequence of edges. The name *differential* lattice is appropriate because, as is obvious, the lattice is formed by partial differentiation (forgetting the differentiation constant).

This lattice is isomorphic to the divisor lattice of the natural numbers. The isomorphism being given by identifying $t_j^{\alpha_j}$ with $p_j^{\alpha_j}$ where p_j is the p-th rational prime. A useful consequence of this fact is that a unique integer is assigned to each monomial. The lattices associated with two distinct monomials intersect in the lattice determined by the monomial associated with the greatet common divisor of the two numbers associated with the original monomials. The complementary result for the lattice-theoretical union of the two lattices and the least common multiple also holds.

In any case, given n and the *exponent vector* $(\beta_1, \dots, \beta_k)$, we can recover the underlying monomial uniquely. We shall use this fact when we refer to the monomial by giving its exponent vector, and abuse language somewhat by calling the exponent vector, the monomial itself. (This differential lattice induces a lattice structure on the Young diagrams of the partitions of n, one which is different from the Young lattice.) This is illustrated for the monomial $t_1 t_2^2 t_3^2$ by

Example 1.



This lattice is useful in organizing the definition of weighted monomials. *Weighting* is a method of systematically supplying the coefficients for a sequence of polynomials indexed by the natural numbers. Here we note that for any isobaric polynomial of degree n the maximal number of monomials is just the number of partitions of n.

In fact, we can regard an isobaric polynomial of degree n as one which is indexed by the shapes for the partitions of n.

2. Weighted Isobaric Polynomials

The concept of *Weighted Isobaric Polynomials* was introduced in [5]. We recall the definition here. We assign a weight (here, integers) to a monomial t^α by first assigning a weight ω_j to the variable t_j for $j = 1, 2, \ldots$, where $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots)$ is a *weight vector*. Then for any monomial in the lattice $\mathcal{L}(t)$ we assign the sum of weights of all nodes that are connected to that node by an edge. For example, after assigning the weights ω_j to the variable t_j in the previous example, the monomial (whose exponent vector is) $(1,0,1)$ gets the weight $\omega_1 + \omega_3$, while the monomial $(0,2,0)$ gets the weight ω_2, and, after a rather tedious calculation using the assignment rule, the monomial $(1,2,2)$ gets the weight $6(\omega_1 + 2\omega_2 + 2\omega_3)$. Fortunately, we can avoid this calculation using the following theorem ([5], Theorem 1).

Theorem 2.1. *Given a weight vector* $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots)$ *the weight assigned to the monomial whose exponent vector is* $(\alpha_1, \ldots, \alpha_k)$ *is*

$$A_\alpha = \binom{\sum \alpha_j}{\alpha_1 \ldots \alpha_k} \frac{\sum_j \alpha_j \omega_j}{\sum_j \alpha_j}. \tag{2.1}$$

Thus any weighted isobaric polynomial is of the form $P_{n,\boldsymbol{\omega}} = \sum_\alpha A_\alpha t^\alpha$, where α ranges over the partitions of n. Each weight vector determines a unique sequence of WIPs. Two such sequences are $\{F_n\}$ and $\{G_n\}$; the coefficients for the F-sequence are given by $A_\alpha = \binom{\sum \alpha_j}{\alpha_j}$ with weight vector $\boldsymbol{\omega} = (1, \ldots, 1, \ldots)$, i.e., $\omega_j = 1$ for all j, and for the G-sequence $A_\alpha = \dfrac{(\sum_j \alpha_j - 1)!}{\prod_j \alpha_j!} n$, with weight vector $\boldsymbol{\omega} = (1, 2, \ldots, k, \ldots)$, i.e., $\omega_j = j$, for all j. It is easily seen that this assignment follows from Theorem 2.1. We call the weight vector of the F-sequence, the *unit weight vector*, and the the G-sequence, *the natural weight vector*.

In [5], Theorem 2.3, it is shown that the sequences of weighted isobaric polynomials form a free $\mathbb{Z}$-module where addition is defined as addition of weight vectors, that is the sum of two sequences of weights $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ is the sequence of weight $\boldsymbol{\omega}'' = \boldsymbol{\omega} + \boldsymbol{\omega}'$. It is also shown in that same paper that isobaric reflects of hook Schur polynomials (i.e., the Schur polynomials determined by hook Young diagrams) are in the WIP-module. (The weight of the hook reflect determined by the hook diagram $(n-r, 1^r)$ is $(-1)^r(0, \ldots, 0, 1, 1, \ldots)$). The hook reflects in fact form a basis for the WIP- module. As an application of the WIP-module structure we have the following isobaric version of a well-known theorem of symmetric functions

Theorem 2.2. $G_n = \sum_{r=0}^n (-1)^r H_r$, *where* H_r *is the hook reflect induced by the shape* $(n-r, 1^r)$.

For, clearly the sum of the alternating sum of the n-hook weights is the weight of G_n. □

The symmetric polynomial version of this is the statement that a complete symmetric polynomial is an alternating sum of Schur hooks.

3. The kernel of $\mathfrak{T}_m$

We now turn our attention to the linear operator $\mathfrak{T}_m$ and find that for certain choices of the parameter m, the F-sequence and the G-sequence belong to the kernel of $\mathfrak{T}_m$.

Theorem 3.1 ([4],Theorem 4). *$\mathfrak{T}_m(F_n) = (D_{11} - \sum_j t_j D_{2j} - mD_2)(F_n) = 0$ when $m = 2$, and $\mathfrak{T}_m(G_n) = (D_{11} - \sum_j t_j D_{2j} - mD_2)(G_n) = 0$ when $m = 1$.*

This theorem will follow from Theorem 3.2 below. Theorem 3.1 tells us that the F- and G-sequences are solutions to the operator equation when the parameter is $m = 1$ in the case of the G-polynomials and $m = 2$ in the case of the F-polynomials, but it turns out that these solutions are determined by other more basic solutions which, while dependent on the weights of the F- and G-sequences, are not themselves WIPs. We shall refer to these polynomials as *satellites.* Their definition depends upon the concept of *string* which we shall define below, and from this the definition of satellite.

Roughly speaking, a *string* associated with an isobaric polynomial of isobaric degree n is an (ordered) subset of monomials that occur in the isobaric polynomial, but , *forgetting coefficients.* They are just the parts of the polynomial that can be represented by the exponent vectors (see Section 1), and, in fact, that is just the notation that we shall use to represent them. The set of strings associated to an isobaric polynomial will be identical for all polynomials of the same isobaric degree.

The upshot will be that for each F_n and $m = 2$ there is a sequence of satellites, themselves isobaric polynomials, $S_1^{F_n}, \dots, S_u^{F_n}$, such that $F_n = \sum_{j=1}^u S_j^{F_n}$ and $\mathfrak{T}_2(S_j^{F_n}) = 0$. Similarly, for each G_n, and $m = 1$, there is a sequence of satellites $S_1^{G_n}, \dots, S_u^{G_n}$, such that $G_n = \sum_{j=1}^u S_j^{G_n}$ and $\mathfrak{T}_1(S_j^{G_n}) = 0$.

DEFINITION (Strings)

Let $(\alpha_1, ..., \alpha_k), k > 1$, be the exponent vectors associated with an isobaric polynomial of isobaric degree n.

We first choose exponent vectors of the following two kinds:

1. vectors of type $(0, \alpha_2, \alpha_3, \dots, \alpha_k)$, where $\alpha_3, \dots, \alpha_k$ is a fixed $(k-2)$-tuple and α_2 is largest second element with respect to this condition. We call such a vector an *even string generator.*
2. vectors of type $(1, \alpha_2, \alpha_3, \dots, \alpha_k)$ where $\alpha_3, \dots, \alpha_k$ is a fixed $(k-2)$-tuple and α_2 is largest second element with respect to this condition. We call such a vector an *odd string generator*

Then we select sequences of vectors of one of the two following forms.

$$\begin{array}{rl}
(0, \alpha_2, \alpha_3, \dots, \alpha_k) & (1, \alpha_2, \alpha_3, \dots, \alpha_k) \\
(2, \alpha_2 - 1, \alpha_3, \dots, \alpha_k) & (3, \alpha_2 - 1, \alpha_3, \dots, \alpha_k) \\
\dots & \dots \\
(2j, \alpha_2 - j, \alpha_3, \dots, \alpha_k) & (2j+1, \alpha_2 - j, \alpha_3, \dots, \alpha_k) \\
\dots & \dots \\
(2\alpha_2, 0, \alpha_3, \dots, \alpha_k) & (2\alpha_2 + 1, 0, \alpha_3, \dots, \alpha_k)
\end{array}$$

$$j = 0, 1, \dots, \alpha_2.$$

The left-hand column of vectors is an *even string* with the top vector as *(even) string generator*. In a similar way, the right-hand column of vectors is an *odd string* with the top vector as *(odd) string generator*. A string is generated by starting with the string generator and increasing the first entry by 2 at each stage and decreasing the second entry by 1 at each stage until the second entry becomes 0. For example, if (1,3,1) is a string generator, then the ordered set of vectors

$$\{(1,3,1),(3.2.1),(5,1,1),(7,0,1)\}$$

is a(n) (odd) string for any isobaric polynomial of isobaric degree 10.

It is not difficult to see that for a given n all of the exponent vectors that arise from the partition of n occur in some even or odd string (just reverse the procedure for constructing a string, beginning with an arbitary exponent vector). Thus, every isobaric polynomial is just the sum of its strings with "remembered" coefficients. A satellite belonging to a given isobaric polynomial is just a string belonging to that polynomial with the coefficients of that polynomial restored. A satellite belongs to a particular isobaric polynomial; a string belongs to a class of isobaric polynomials.

In particular, for a sequence of weighted isobaric polynomials, each polynomial is just the weighted sum of its strings with remembered coefficients. Theorem 3.1 will follow from this fact.

The following example will illustrate these ideas: the (three) strings that *belong* to F_4 , where $F_4 = t_1^4 + 3t_1^2t_2 + t_2^2 + 2t_1t_3 + t_4$, are

$$\{(0,2,0,0),(2,1,0,0),(4,0,0,0)\};\{(1,0,1,0)\};and\{(0,0,0,1)\},$$

two even strings and an odd string. The three satellites determined in F_4 by these strings are $t_1^4 + 3t_1^2t_2$; $2t_1t_3$; and t_4.

It is clear that the strings reflect the truncations of the isobaric polynomial obtained by deleting the variables t_j for the j's from a certain j onward.

It is also clear that differential operators induce mappings of the differential lattice into itself. This fact will be of use in what follows in motivating and understanding proofs of the theorems.

And finally, we shall say that two elements in a string are *adjacent* if they are adjacent in the ordering of the string.

We are now ready to state and prove

Theorem 3.2.
(1) If S^F is a satellite belonging to F, then $\mathfrak{T}_2(S^F) = 0$.
(2) If S^G is a satellite belonging to G, then $\mathfrak{T}_1(S^G) = 0$.

This theorem will follow from

Lemma 3.3. *a). $(2j+2, \alpha_2 - j - 1, \alpha_3, \ldots, \alpha_k)$ and $(2j, \alpha_2 - j, \alpha_3, \ldots, \alpha_k)$ are adjacent elements in the even string generated by $(0, \alpha_2, \alpha_3, \ldots, \alpha_k)$ and the coefficient of $D_{11}(2j+2, \alpha_2 - j - 1, \alpha_3, \ldots, \alpha_k)$ equals the coefficient of $-(\mathfrak{T}_m - D_{11})(2j, \alpha_2 - j, \alpha_3, \ldots, \alpha_k)$ whenever the weight vector is $(1,1,\ldots,1,\ldots)$ and $m = 2$, or the weight vector is $(1,2,\ldots,k,\ldots)$ and $m = 1$. D_{11} applied to the string generator is 0 and $(\mathfrak{T}_m - D_{11})$ applied to the last element in the string is also 0.*

b). $(2j+3, \alpha_2 - j - 1, \alpha_3, \ldots, \alpha_k)$ and $(2j+1, \alpha_2 - j, \alpha_3, \ldots, \alpha_k)$ are adjacent elements in the odd string generated by $(1, \alpha_2, \alpha_3, \ldots, \alpha_k)$ and the coefficient of $D_{11}(2j+3, \alpha_2 - j - 1, \alpha_3, \ldots, \alpha_k)$ equals the coefficient of $-(\mathfrak{T}_m - D_{11})(2j+1, \alpha_2 - j, \alpha_3, \ldots, \alpha_k)$ whenever the weight vector is $(1,1,\ldots,1,\ldots)$ and $m = 2$, or the weight vector is

$(1, 2, \ldots, k, \ldots)$ and $m = 1$. D_{11} applied to the string generator is 0 and $(\mathfrak{T}_m - D_{11})$ applied to the last element in the string is also 0.

Proof of Lemma That the elements mentioned in the lemma belong to the string and are adjacent is obvious. The fact that the first and last elements of the string are mapped to 0 by the operators D_{11} and $(\mathfrak{T}_m - D_{11})$ as claimed is also obvious. We shall prove then that the coefficients of the elements $D_{11}(2j+2, \alpha_2 - j - 1, \alpha_3, \ldots, \alpha_k)$ and $(\mathfrak{T}_m - D_{11})(2j, \alpha_2 - j, \alpha_3, \ldots, \alpha_k)$ are negatives of one another.

By Theorem 2.1 we have that

$$A_{s_{2j+2}} = \frac{(\sum_{i=2}^{k} \alpha_i + j)!}{(2j+2)!(\alpha_2 - j - 1)! \prod_{i=3}^{k} \alpha_i!}[(2j+2)\omega_1 + (\alpha_2 - j - 1)\omega_2 + \sum_{i=3}^{k} \alpha_i \omega_i] \tag{3.1}$$

$$A_{s_{2j}} = \frac{(\sum_{i=2}^{k} \alpha_i + j - 1)!}{(2j)!(\alpha_2 - j)! \prod_{i=3}^{k} \alpha_i!}[(2j)\omega_1 + (\alpha_2 - j)\omega_2 + \sum_{i=3}^{k} \alpha_i \omega_i], \tag{3.2}$$

where $s_{2j+2} = (2j+2, \alpha_2 - j - 1, \ldots, \alpha_k)$ and $s_{2j} = (2j, \alpha_2 - j, \ldots, \alpha_k)$. , and where $A_{s_{2j+2}}$ and $A_{s_{2j}}$ are the coefficients of these elements determined by the weights (Theorem 2.1). The coefficient due to D_{11} applied to s_{2j+2} is

$$(2j+2)(2j+1), \tag{3.3}$$

and the coefficient due to $\mathfrak{T}_m - D_{11}$ applied to s_{2j} is

$$(\alpha_2 - j)(\sum_{i=2}^{k} \alpha_i + j + m - 1). \tag{3.4}$$

Multiplying equation (3.1) by (3.3) and equation (3.2) by (3.4) and using the values given by the hypothesis of the lemma for m and for the weight vector and comparing gives the result.

It is useful to record the last steps in the computation beginning just before the hypotheses on m and the weight vectors are applied. We have this expression

$$\begin{aligned} &(\sum_{i=2}^{k} \alpha_i + j)((2j+2)\omega_1 + (\alpha_2 - j - 1)\omega_2 + \sum_{i=3}^{k} \alpha_i \omega_i) \\ &-(\sum_{i=1}^{k} \alpha_i + j + m - 1)((2j)\omega_1 + (\alpha_2 - j)\omega_2 + \sum_{i=3}^{k} \alpha_i \omega_i) \end{aligned} \tag{3.5}$$

Letting $m = 1$ gives $2\omega_1 - \omega_2 = 0$, after applying the hypothesis on the weights, which gives the result required no matter what the weights ω_j are for $j > 2$. Thus we have proved more in this case, that is, we have infinitely many WIP sequences as solutions.

Letting $m = 2$ gives the expression

$$(\sum_{i=1}^{k} \alpha_i + j)(2\omega_1 - \omega_2) - ((2j)\omega_1 + (\alpha_2 - j)\omega_2 + \sum_{i=3}^{k} \alpha_i \omega_i), \tag{3.6}$$

but now we need our weight hypothesis on all of the weights to achieve the cancellation, thus this expression is 0 if we assume that

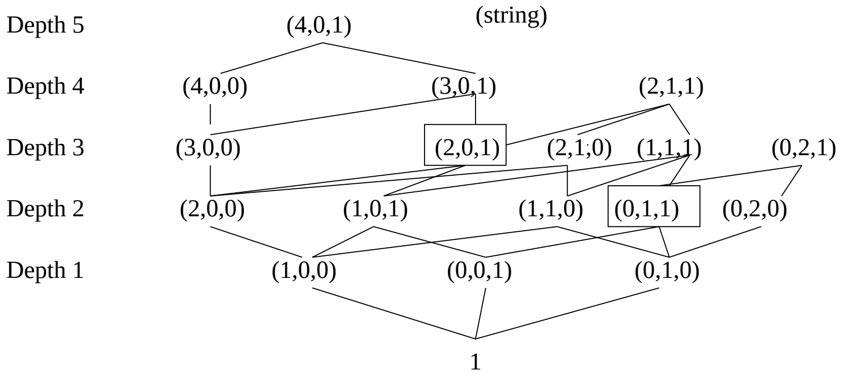
$\omega_j = \omega_1$, for all j.

The proof of part b). is similar to that of part a). and will be omitted. □

But then Theorem 3.2 now follows from the lemmas. Theorem 3.1 follows from Theorem 3.2 by linearity.

It is an interesting consequence of the proof that the lattices of the string elements intersect for the first time exactly at the nodes determined by D_{11} operating on the string. We give an example.

Example 2. *Consider the string generated by (0,2,1), $n = 7$, $k = 3$. The lattices are given by*



In this case the intersection nodes are $(2, 0, 1)$ and $(0, 1, 1)$. (Note that this is consistent with the claim made in 1.1 that intersections of sublattices are determined by gcd's associated with relevant nodes of the divisor lattice of the natural numbers.) The string consists of the three nodes $(4, 0, 1)$, $(2, 1, 1)$ and the string generator is $(0, 2, 1)$.

It is also the case that the intersection nodes again form a string. This time for polynomials of degree $n - 2$. However, these strings do not inherit the weighting of the string of degree n.

4. Other solutions

We stress here that what we mean by a solution of the differential operator is the entire sequence of WIPs determined by a particular weight vector $\boldsymbol{\omega}$; calling such a sequence $P_{\boldsymbol{\omega}}$, $P_{\boldsymbol{\omega}} = \{P_{n,\boldsymbol{\omega}}\}$,we have as solutions the polynomials generated by the strings of WIP-solutions. We claim that the WIP sequences of solutions of the PDE $\mathfrak{T}_1 = 0$ (that is, for $m = 1$) are exactly those solutions generated by linearity from the strings in which $2\omega_1 = \omega_2$ with the ω_j arbitrary for $j > 2$, but fixed throughout the string; G-polynomials, for example.

When $m = 2$, the solutions of $\mathfrak{T}_2 = 0$ consist just of the scalar multiples of the F-polynomials. The kernel of the operator operating on the WIP-module is 0 when $m \neq 1, 2$.

The G-polynomials are one of many WIP-solutions of the operator equation when $m = 1$; but when $m = 2$, the only WIP solutions are just the scalar multiples of the F-polynomials. We prove these assertions now.

Proposition 4.1. *Let $\boldsymbol{\omega}$ be a weight vector and $\{P_{n,\boldsymbol{\omega}}\}$ be a sequence of solutions of $P_{n,\boldsymbol{\omega}} \in$ WIP-module, then either*

(1) $m = 1$ *and* $2\omega_1 = \omega_2$ *or*
(2) $m = 2$ *and* $\omega_1 = \omega_2$.

Proof It is only necessary to look at the second and third terms of the sequence; namely, at

$$P_{2,\boldsymbol{\omega}} = \omega_1 t_1^2 + \omega_2 t_2,$$
$$P_{3,\boldsymbol{\omega}} = \omega_1 t_1^3 + (\omega_1 + \omega_2) t_1 t_2 + \omega_3 t_3.$$

Requiring that $P_{2,\boldsymbol{\omega}}$ satisfies the operator equation implies that $m\omega_2 = 2\omega_1$; requiring that $P_{3,\boldsymbol{\omega}}$ satisfies yields $m(\omega_1 + \omega_2) = (5\omega_1 - \omega_2)$.
Setting the two values equal and solving the resultant quadratic in $\mathbb{Z}$ gives the two possibilities $2\omega_1 = \omega_2$, or $\omega_1 = \omega_2$. In either case, the trivial weight ω is a possibility. Solving for m in each case gives $m = 1$ and $m = 2$ or the trivial weight for any m, respectively. And we know that the first two cases are realized with the G-polynomial sequence and the F-polynomial sequence respectively. □

We summarize our discussion of solutions of the operator so far in the following Corollary of Proposition 4.1.

Corollary 4.2. *In case (1) of Proposition 4.1, the condition* $2\omega_1 = \omega_2$ *characterizes the solutions of the operator equation* $\mathfrak{T}_1 = 0$. *In particular, the first two components of the weight vector completely determine the kernel.*

In case (2) of Proposition 4.1, if we ask for solutions generated by string solutions then $\omega_1 = \omega_2 = a$ *implies that* $\omega_j = a$ *for all* j. *That is, all such WIP-solutions are of the form* aF_n, $n \in \mathbb{N}$, *and all solutions are exactly those generated in the WIP-module by F-strings.*

Proof The proof consists of looking at the proof of Theorem 3.2 more carefully and noting that in light of Proposition 4.1 (1), the cancellation occurs independently of the choice of ω_j for $j > 2$, while in the case of Proposition 4.2 (2), the proof arrives at the equation $\sum_j a\alpha_j = \sum_j \alpha_j \omega_j$ with $\omega_1 = \omega_2 = a$, which must hold for all exponent vectors α and for a fixed weight vector ω; thus $\omega_j = a$ for all j. □

So now we come to the three questions posed in the introduction. It turns out that we shall be able to answer these questions completely once we have answered the third one. So our aim is to prove

Theorem 4.3. *The operator* $D_{11} - \sum_j a_j t_j D_{i,j} - mD_2$, *where* $a_j \in \mathbb{Z}$, *has a nontrivial WIP-sequence solutions only when* $a_j = 1$ *and* $m = 1$ *or* $m = 2$, *where the* a_j *and* m *are assumed to be arbitrary real numbers, not all zero (Though, they could be taken from any field of characteristics* 0 *as far as the proof is concerned).*

The statement of this theorem makes clear what we have chosen to mean by a generalized operator. When one tries to find other second order, linear partial differential equations that have sufficient resemblance to the one at hand, the lack of left-right symmetry among the partitions of n as n increases becomes more apparent. This is due to the fact that 1's will appear in the decompositions of n many times, but n itself can appear only once; small numbers have the advantage over big ones. This is reflected in the futility of trying to find new PDEs by varying the suffixes of the operators, $D_{i,j}$. However, a tack that appears promising is to provide $\mathfrak{T}_m$ with arbitrary (real) coefficients. Thus, we want to ask what is the kernel of any operator of the sort $D_{11} - \sum_j a_j t_j D_{2,j} - mD_2$, a_j arbitrary (real) scalars? (Here we assume that the coefficient of D_{11} is not 0, so we can, without loss of generality, assume that it is 1.) By the way, the resemblance of the operator equation $\mathfrak{T}_m = 0$ to the "Newton identity" satisfied by the WIP-polynomials (see Theorem 4.1 [5]) is striking, and probably significant, though the anomolous role of the D_2-term is puzzling.

To prove Theorem 4.3 we will first show that Theorem 3.2 generalizes, that is that a weighted isobaric polynomial is a solution of the operator equation if and only if all of its satellites are solutions. To do this, we make use of the differential lattice and the fact mentioned in section 3., namely, that differential operators are lattice injections. In particular, we observe that D_{11} and $(\mathfrak{T}_m - D_{11})$ map any element of a string either to an element of the union of the lattices of the string elements or to the empty element, which we call 0. The figure of Example 2 makes this point clear and obvious. We enshrine this fact in

Lemma 4.4. *Let S be a string belonging to $P_{n,\boldsymbol{\omega}}$, and let s be an element of S then $D_{11}(s)$ and $(\mathfrak{T}_m - D_{11})(s) \in$ the union of the lattices of the string elements; call this the lattice of the string.*

Proof This is clear from the definition of the differential lattice. □

Theorem 4.5. *$P_{n,\boldsymbol{\omega}}$is a solution of $\mathfrak{T}_m = 0$ if and only if the satellites belonging to $P_{n,\boldsymbol{\omega}}$ are solutions.*

Proof Clearly, since $P_{n,\boldsymbol{\omega}}$ is just the sum of its satellites, we need only prove the necessity. So suppose that $P_{n,\boldsymbol{\omega}}$ is a solution of $\mathfrak{T}_m = 0$, but then the theorem follows from Lemma 4.4 and the proof and statement of lemma 3.3. (We shall later refer to that proof as a *domino* proof.) □

Next, we note that the depths of elements in a string S form a strictly monotonically increasing sequence. It is also easy to see that $\mathfrak{T}_m(S)$ is also a string. From the fact that D_{11} and $\mathfrak{T}_m - D_{11}$ each have exactly one monomial as an image (Lemma 3.3), we have, using Lemma 4.4,

Lemma 4.6. *A satellite is a solution of $\mathfrak{T}_m = 0$ if only if the proof is the "domino" proof used in the proof of Theorem 3.2.* □

The proof of Proposition 4.1 contains the following fact which, together with its proof, also holds in the generalized operator case.

Lemma 4.7. *If $P_{2,\omega}$ satisfies the generalized operator equation, then $m\omega_2 = 2\omega_1$.* □

Proof of Theorem 4.3 Now, it is clear that if the satellites of $P_{n,\boldsymbol{\omega}}$ satisfy the operator equation, then so does $P_{n,\boldsymbol{\omega}}$. So let us suppose, conversely, that $P_{n,\boldsymbol{\omega}}$ satisfies the operator equation. We consider the "domino" proof used in the proof of Theorem 3.2. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k)$ be an arbitrary element in a string S. If α is the only element of the underlying string, then clearly it satisfies the operator equation. So suppose that α is not an element of least depth, that is, it is not the generating element of the string. In this case, there is an element of depth one less than the depth of α. Let us suppose that $D_{11}(A_\omega \alpha) = (\mathfrak{T}_m - D_{11})(B_\omega \beta)$, that is suppose that the "domino" proof applies. (Note that if α is an element of greatest depth, then $D_{11}(\alpha) = 0$.) The picture looks like this:

$$\begin{tikzcd}
(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k) = \alpha \arrow[d, dash] & \\
(\alpha_1 - 1, \alpha_2, \alpha_3, ..., \alpha_k) \arrow[d, dash] & (\alpha_1 - 2, \alpha_2 + 1, \alpha_3, ..., \alpha_k) = \beta \arrow[dl, dash] \\
(\alpha_1 - 2, \alpha_2, \alpha_3, ..., \alpha_k) &
\end{tikzcd}$$

Recalling the proof of Theorem 3.2 at this point, we needed to equate the product of the coefficient of A_ω of α and the coefficient of $D_{11}(\alpha)$ with the product of the coefficient B_ω of β and $(\mathfrak{T}_m - D_{11})(\beta)$. The new ingredient here is that $(\mathfrak{T}_m - D_{11})(\beta) = (\alpha_2 + 1)(\sum_j a_j \alpha_j + m - 2a_1)$, due to the new coefficients of $\mathfrak{T}_m - D_{11}$. After making the calculation indicated above and allowing the dust to settle, this gives

$$(\sum_{j=1}^{k} \alpha_j - 1)(\sum_{j=1}^{k} \alpha_j \omega_j) = (\sum_{j=1}^{k} \alpha_j \omega_j + \omega_2 - 2\omega_1)(\sum_{j=1}^{k} a_j \alpha_j + m - 2a_1) \tag{4.1}$$

as a necessary condition for the generalized operator to have a solution.

We shall assume throughout that P_ω is not trivial, that is, that $\boldsymbol{\omega} \neq 0$. Equation (4.1) can be rewritten as

$$(\sum_{j=1}^{k} \alpha_j - 1)(\sum_{j=1}^{k} \alpha_j \omega_j) - (\sum_{j=1}^{k} \alpha_j \omega_j)(\sum_{j=1}^{k} a_j \alpha_j + m - 2a_1) \tag{4.2}$$

$$-(\omega_2 - 2\omega_1)(\sum_{j=1}^{k} a_j \alpha_j) = (\omega_2 - 2\omega_1)(m - 2a_1)$$

The left hand side of (4.2) depends on α , which is a variable, while the right hand side depends only on the choice of $\boldsymbol{\omega}$ and the constants a_1 and m. Hence the left hand side and the right hand side of (4.2) are independently 0. And hence, either $\omega_2 - 2\omega_1 = 0$ or $m - 2a_1 = 0$.

In the first case, we have then that $\omega_2 = 2\omega_1$ and, by Lemma 4.8, $m = 1$. The left hand side of (4.2) becomes

$$(\sum_{j=1}^{k} \alpha_j \omega_j)(\sum_{j=1}^{k} \alpha_j - \sum_{j=1}^{k} a_j \alpha_j + 2 - 2a_1) = 0. \tag{4.3}$$

$(\sum_{j=1}^{k} \alpha_j \omega_j) = 0$ implies that $\boldsymbol{\omega} = 0$, that is, the solution is trivial.

Thus $(\sum_{j=1}^{k} \alpha_j - \sum_{j=1}^{k} a_j \alpha_j + 2 - 2a_1 = 0)$ and so we have as above that $\sum_{j=1}^{k} \alpha_j - \sum_{j=1}^{k} a_j \alpha_j = 0$ and $2 - 2a_1 = 0$. And so we have that

$$a_1 = 1 \text{ and } \sum_{j=1}^{k} \alpha_j = \sum_{j=1}^{k} a_j \alpha_j. \tag{4.4}$$

From these equations we have that $a_j = 1$ for $j = 1, \ldots, k$. This is just the case of the original operator for which the G-polynomials were solutions.

In the second case, from $m - 2a_1 = 0$, we have $m = 2a_1$ and from this and (4.1) we have

$$\sum_{j=1}^{k} \alpha_j \omega_j (\sum_{j=1}^{k} \alpha_j - \sum_{j=1}^{k} \alpha_j a_j - 1) = (2\omega_1 - \omega_2)(\sum_{j=1}^{k} \alpha_j - j a_j). \tag{4.5}$$

Consider the monomial $\omega_n t_n$. It follows from the definition of a string that $\omega_n t_n$ is a satellite, or, in the case that $n = 2$, is the generator of a two element satellite, so we apply Theorem 4.5. Here $\alpha_n = 1$ and $\alpha_j = 0$ otherwise. From (4.5) we arrive at

$$\omega_n a_n = (2\omega_1 - \omega_2)(a_n). \tag{4.6}$$

We may assume that $a_n \neq 0$ for some n. Then, when $n = 2$ we have that $\omega_2 = \omega_1$ for all n, so that $\omega_n = 2\omega_1 - \omega_2$ is constant for all n. In particular, when $n = 2$ we have that $\omega_2 = \omega_1$, and thus $\omega_n = \omega_1$ for all n. So, in particular, when $n = 2$, $\omega_2 = \omega_1$ which, in turn, implies that $\omega_n = \omega_1$ for all n. From (4.1) it then follows that $\omega_1(\sum_{j=1}^{k} \alpha_j - 1)(\sum_{j=1}^{k} \alpha_j) = (\sum_{j=1}^{k} \alpha_j - 1)(\sum_{j=1}^{k} a_j \alpha_j)\omega_1$. If $\boldsymbol{\omega} \neq 0$, that is, if the solution is not the trivial solution, we have that $\sum_{j=1}^{k} \alpha_j = \sum_{j=1}^{k} a_j \alpha_j$ from which it follows that $a_j = 1$ for $j = 1, \ldots, k$. Moreover, $m = 2$; and this is just the case of solutions generated by the strings of F-polynomials and the original operator. □

Remark : In the case that the sequence P_ω is a solution of the operator equation and if $\omega_2 = 0$ (and thus $\omega_1 = 0$), then either $\omega_j = 0$ for all j, or $m = 1$.

.

This follows easily by applying Theorem 4.5, Lemma 4.7 and the assumptions to strings generated first by$(0, 1, \ldots, 1, 0, 0)$ and then by $(1, 1, \ldots, 1, 0, 0, 0)$, with the last non-zero entry in each generator being the exponent of t_n for the same n in each case.

We have then that the answer to question (3) is that the only WIP solutions for the generalized operator equation occur when $a_j = 1$ for all $j \in \mathbb{N}$, and $m = 1$ or 2. Thus the generalized operator has a zero kernel except in the case we started with, thus generalizing the operator does not produce new solutions. Clearly, we have also answered question (2); allowing m to vary beyond 1 and 2, in fact, over any field of characteristic 0, produces no new solutions. The answer to question (1), we learn here, is yes and no. If $m = 2$, then the answer is unique up to a scalar multiple, that is all WIP-solutions are scalar multiples of the F-sequence; but if $m = 1$, then not only are scalar multiples of the G-sequence solutions, but also so is the sequence $P_{\boldsymbol{\omega}}$ anytime that $2\omega_1 = \omega_2$, the remaining weights being arbitrary. However, we also have satellite solutions which get their life from the WIPs, but are not themselves WIPs.

It is tempting to think that a weight vector for an initial satellite of $P_{n,\boldsymbol{\omega}}$ (i.e., the string whose terminal element is $(n, 0, \alpha_3, \ldots, \alpha_k)$) might be reweighted as $(\omega_1, \omega_2, 0, \ldots, 0, \ldots)$, while $\boldsymbol{\omega} = (\omega_j)$, where ω_j is different from 0 for at least one $j > 2$; that is, that we can choose a new weight vector for the satellite by keeping the first two entries of the original weighting the same, and let all of the other entries be 0. The following example shows what goes wrong here.

Example 3. *Consider* $P_{n,\boldsymbol{\omega}} = \omega_1 t_1^4 + (2\omega_1 + \omega_2)t_1^2 t_2 + \omega_2 t_2^2 + (\omega_1 + \omega_3)t_1 t_3 + \omega_4 t_4$. *The strings are*:

$$\begin{array}{lll} \textit{Initial String} & & \\ \left\{\begin{array}{l}(0,2,0,0)\\(2,1,0,0)\\(4,0,0,0)\end{array}\right. & \left\{(1,0,1,0)\right. & \left\{(0,0,0,1)\right. \end{array}$$

Try weight vector $(\omega_1, \omega_2, 0, 0)$. But now by Theorem 3.1, if the initial string is a WIP, then $(1,0,1,0)$, that is the monomial $t_1 t_3$, has coefficient $\omega_1 + \omega_3$, while $t_1 t_3$ has weight ω_1 in the new weighting—recall, we have to assign a weight to each of the monomials induced by a partition of n, thus $t_1 t_3$ would appear in the initial string if $\omega_1 \neq 0$. This contradiction would appear more generally. We omit the proof.

It is also interesting to note the rather special companionable role that the F-sequences and G-sequences play among the isobaric polynomials, especially among the WIPs. In addition to the properties shown in this paper, we have, for example, that the G's are related to the F's by partial differentiation as follows: $\partial/\partial t_j(G_n) = F_{n-j}$. In general, $\partial/\partial t_j(P_{\boldsymbol{\omega}})$ is not a WIP, in fact, there is good reason to believe that this is the only case. We pursue this observation in a later paper.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, NORTH YORK, ONT., M3J 1P3, CANADA AND SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN, 55455, USA

E-mail address: `machenry@mathstat.yorku.ca`

E-mail address: `tudose@math.umn.edu`