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MONOTONIC SEQUENCES RELATED TO ZEROS OF BESSEL FUNCTIONS

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To the memory of Luigi Gatteschi

ABSTRACT. In the course of their work on Salem numbers and uniform distribution modulo 1, A. Akiyama and Y. Tanigawa proved some inequalities concerning the values of the Bessel function J_0 at multiples of π, i.e., at the zeros of $J_{1/2}$. This raises the question of inequalities and monotonicity properties for the sequences of values of one cylinder function at the zeros of another such function. Here we derive such results by differential equations methods.

1. Introduction

As far back as 1950 [8], Luigi Gatteschi was interested in the approximation of Bessel functions and their zeros. Later, he contributed greatly to the use of Bessel functions as approximants for other functions; see, e.g., [9, 10] and references. He often used differential equations methods. Since the trigonometric functions are related to the Bessel functions by

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x,$$

it is natural to investigate the question of whether various sequences which are constant in the trigonometric case might be monotonic in the case of general order. Such questions are studied here. They are more elementary than those dealt with by Gatteschi. The element of commonality is the use of differential equations methods.

This study is motivated by the result

$$|J_0(2k\pi)| \le 1/(\pi\sqrt{2k}), \; k = 1, 2, \ldots. \tag{1.1}$$

of A. Akiyama and Y. Tanigawa [1, Lemma 2], used in the course of their work on Salem numbers and uniform distribution modulo 1.

We are dealing here with the values of the Bessel function J_0 at multiples of π, i.e., at the zeros of $J_{1/2}$. This raises the question of inequalities and monotonicity properties for the sequences of values of one cylinder function at the zeros of another such function. For example, we show (Corollary 5.5) that for $0 \le \nu < \frac{1}{2}$, $(-1)^k\sqrt{k}J_\nu(k\pi)$ increases to its limiting value $(\sqrt{2}/\pi)\sin((1-2\nu)\pi/4)$ as k $(= 1, 2, \ldots)$ increases. In particular, with $\nu = 0$ this gives

$$0 < (-1)^k J_0(k\pi) < \frac{1}{\pi\sqrt{k}}, \quad k = 1, 2, \ldots, \tag{1.2}$$

generalizing (1.1).

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2. Transformations of differential equations

We consider the differential equation

$$y'' + f(x)y = 0, \quad a < x < \infty, \tag{2.1}$$

where f is continuous on (a,∞), and the equation is nonoscillatory [11, p. 351] at a. Under these conditions, the equation has a principal solution at a [11, p. 355], i.e., a solution $y_1(x)$ such that for every solution $y(x)$ linearly independent of y_1, we have

$$\lim_{x\to a^+} y_1(x)/y(x) = 0. \tag{2.2}$$

Let y_2 be a solution of (2.1) such that the Wronskian

$$W(y_1,y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \equiv 1, \tag{2.3}$$

and let

$$p(x) = y_1^2(x) + y_2^2(x) \to 1, \tag{2.4}$$

as $x\to\infty$. We suppose throughout that, for a fixed c, $a < c < \infty$,

$$\int_a^c \frac{du}{p(u)} = \lim_{\epsilon\to 0^+}\int_{a+\epsilon}^c \frac{du}{p(u)} < \infty. \tag{2.5}$$

It is well known that equation (2.1) can be transformed to the trigonometric equation $u''(t)+u(t)=0$. The changes of variables required are described, for example in [13, Lemma 2.3], where the interest, like here, is in the application to zeros of Bessel functions. Letting $y(x) = [p(x)]^{1/2}u(t)$, $x'(t) = p(x)$ (2.1) becomes

$$u''(t) + u(t) = 0, \tag{2.6}$$

where the prime now denotes differentiation with respect to t. Hence the general solution of (2.1) may be written as

$$y(x) = A[p(x)]^{1/2}\sin\left(\int_a^x \frac{dt}{p(t)} + \alpha\right), \tag{2.7}$$

where A and α $(0\le\alpha<\pi)$ are arbitrary. The solutions y_1, y_2 are unique up to their replacement by

$$\cos\alpha\, y_1(x) - \sin\alpha\, y_2(x),\ \sin\alpha\, y_1(x) + \cos\alpha\, y_2(x)$$

in the sense that this leaves $W(y_1,y_2)$ and $p(x)$ unchanged. So, from here on, we will take

$$y_1(x) = [p(x)]^{1/2}\sin\left(\int_a^x \frac{dt}{p(t)}\right),\ y_2(x) = -[p(x)]^{1/2}\cos\left(\int_a^x \frac{dt}{p(t)}\right). \tag{2.8}$$

We use $y(x,\alpha)$ and $y^*(x,\alpha) = y(x,\alpha+\pi/2)$ for the solutions

$$y(x,\alpha) = \cos\alpha\, y_1(x) - \sin\alpha\, y_2(x) = [p(x)]^{1/2}\sin\left(\int_a^x \frac{dt}{p(t)} + \alpha\right), \tag{2.9}$$

$$y^*(x,\alpha) = -\sin\alpha\, y_1(x) - \cos\alpha\, y_2(x) = [p(x)]^{1/2}\sin\left(\int_a^x \frac{dt}{p(t)} + \alpha + \pi/2\right). \tag{2.10}$$

The zeros of $y(x,\alpha)$ on (a,∞) are the numbers x_k for which

$$\int_a^{x_k} \frac{dt}{p(t)} = k\pi - \alpha, \quad k = 1,2,\ldots. \tag{2.11}$$

For consistency in notation, we take $x_0 = a$, when $\alpha = 0$.

At a zero x_k of $y(x,\alpha)$ we have

$$y(x_k,\beta) = (-1)^k \sqrt{p(x_k)} \sin(\beta - \alpha) \tag{2.12}$$

leading to

Remark 2.1. *If $p(x)$ is monotonic as a function of x, so is the sequence $y(x_k,\beta)$.*

Now we consider a second equation

$$Y'' + F(x)Y = 0, \quad a < x < \infty, \tag{2.13}$$

with a similar notation. $Y_1(x)$ and $Y_2(x)$ denote linearly independent solutions of (2.13) with Wronskian 1, such that

$$P(x) = Y_1^2(x) + Y_2^2(x) \to 1, \tag{2.14}$$

as $x \to +\infty$. The general solution of (2.13) may be written as

$$Y(x) = A[P(x)]^{1/2} \sin\left(\int_a^x \frac{dt}{P(t)} + \alpha\right), \tag{2.15}$$

where A and α ($0 \le \alpha < \pi$) are arbitrary. Without loss of generality, we may take

$$Y_1(x) = [P(x)]^{1/2} \sin\left(\int_a^x \frac{dt}{P(t)}\right),\ Y_2(x) = -[P(x)]^{1/2} \cos\left(\int_a^x \frac{dt}{P(t)}\right). \tag{2.16}$$

We use $Y(x,\beta)$ for the solution

$$Y(x,\beta) = \cos\beta\, Y_1(x) - \sin\beta\, Y_2(x) = [P(x)]^{1/2} \sin\left(\int_a^x \frac{dt}{P(t)} + \beta\right). \tag{2.17}$$

The zeros of $Y(x,\beta)$ (on (a,∞)) are the numbers X_k for which

$$\int_a^{X_k} \frac{dt}{P(t)} = k\pi - \beta, \quad k = 1, 2, \dots. \tag{2.18}$$

Again, we take $X_0 = a$, in case $\beta = 0$.

3. General results

Our main result deals with the values of $y(x,\alpha)$ at the zeros of $Y(x,\beta)$.

Theorem 3.1. *Consider the solutions $y(x) = y(x,\alpha)$, $y^*(x) = y(x,\alpha - \pi/2)$ of (2.1) given by (2.9) and (2.10) and a solution $Y(x)$ of (2.13) with successive simple zeros at x_k, x_k^* and X_k ($k = 0, 1, \dots$), respectively on (a,∞). Suppose that*

$$p(x) < P(x),\ a < x < \infty, \tag{3.1}$$

and that

$$x_k \le X_k \le x_{k+1}^*,\ k = 1, 2, \dots. \tag{3.2}$$

Let $y(X_1) > 0$. Then $(-1)^{k+1} y(X_k)/\sqrt{p(X_k)}$ increases to a finite limit as k ($= 1, 2, \dots$) increases. In case (3.2) is replaced by

$$x_{k+1}^* \le X_k \le x_{k+1},\ k = 1, 2, \dots, \tag{3.3}$$

the conclusion is that $(-1)^{k+1} y(X_k)/\sqrt{p(X_k)}$ decreases to its limit as k ($= 1, 2, \dots$) increases. The monotonicities are reversed if the inequality sign in (3.1) is reversed.

Proof. Because of $y(X_1) > 0$ and (3.2), we may use $|y(X_k)| = (-1)^{k+1} y(X_k)$. We have to show that $|\sin A_k| < |\sin A_{k+1}|$, $k = 1, 2, \ldots$, where

$$A_k = \int_a^{X_k} \frac{du}{p(u)} + \alpha, \quad k = 1, 2, \ldots.$$

(3.2) gives

$$k\pi \le A_k \le (k + \frac{1}{2})\pi, \tag{3.4}$$

so the A_k are in x-intervals where $|\sin x|$ is increasing. Also

$$A_{k+1} - A_k = \int_{X_k}^{X_{k+1}} \frac{du}{p(u)} > \int_{X_k}^{X_{k+1}} \frac{du}{P(u)} = \pi, \tag{3.5}$$

the inequality following from (3.1). From (3.4) and (3.5), we get

$$(k+1)\pi \le A_k + \pi < A_{k+1} \le (k + \frac{3}{2})\pi \tag{3.6}$$

Thus we have $|\sin A_{k+1}| > |\sin(A_k + \pi)| = |\sin A_k|$ and the assertion about the increase of $(-1)^{k+1} y(X_k)/\sqrt{p(X_k)}$ is proved.

In case (3.2) is replaced by (3.3), the argument is similar except that (3.4) is replaced by

$$(k + \frac{1}{2})\pi \le A_k \le (k+1)\pi, \tag{3.7}$$

so the A_k are in x-intervals where $|\sin x|$ is decreasing and (3.6) is replaced by

$$(k + \frac{3}{2})\pi \le A_k + \pi < A_{k+1} \le (k+2)\pi, \tag{3.8}$$

so the result is $|\sin A_k| > |\sin A_{k+1}|$.

The proof of the assertion in the last sentence follows on noting that when the inequality in (3.1) is reversed, the same happens to the inequality in (3.5).

4. Preliminary remarks on zeros of Bessel functions

Here we consider the special case of equation (2.1) given by

$$y'' + \left[1 + \frac{\frac{1}{4} - \nu^2}{x^2}\right] y = 0, \tag{4.1}$$

satisfied by $y_1(x) = \sqrt{\pi x/2} J_\nu(x)$ and $y_2(x) = \sqrt{\pi x/2} Y_\nu(x)$, with

$$W(y_1, y_2) = 1,$$

and

$$p(x) = p_\nu(x) = \frac{\pi}{2} x \left[J_\nu^2(x) + Y_\nu^2(x)\right] \to 1, \tag{4.2}$$

as $x \to \infty$.

We use the usual notation

$$\mathcal{C}_\nu(x, \alpha) = \cos\alpha J_\nu(x) - \sin\alpha Y_\nu(x),$$

for cylinder functions, and we use $c_{\nu k}(\alpha)$ for the kth positive zero of $\mathcal{C}_\nu(x, \alpha)$. We also use the usual notations $j_{\nu k}$ and $y_{\nu k}$ for the kth positive zeros of $J_\nu(x)$ and $Y_\nu(x)$.

Á. Elbert and A. Laforgia [5] (see also [3]) have shown how to define a zero $j_{\nu\kappa}$ of continuous rank κ of $\mathcal{C}_\nu(x,\alpha)$ by:

$$j_{\nu\kappa} = c_{\nu k}, \text{ where } \kappa = k - \frac{\alpha}{\pi}, \ \ 0 \le \alpha < \pi.$$

So, for example, when $\nu > -1$, $j_{\nu,k}$ and $j_{\nu,k-\frac{1}{2}}$, $k = 1, 2, \dots$ give the positive zeros of $J_\nu(x)$ and $Y_\nu(x)$, respectively. See also the discussion and Figure 1 in [14], which confirms the results of Elbert and Laforgia [6, Theorem 2.2] that $j_{\nu\kappa}$ is a convex function of κ for $0 \le \nu < \frac{1}{2}$ and a concave function of κ for $\nu > \frac{1}{2}$. It follows from the relation [2]

$$j_{-\nu,\nu+\kappa} = j_{\nu,\kappa}, \ \nu \ge 0, \ \kappa > 0,$$

that $j_{\nu\kappa}$ is also a convex function of κ on its domain of definition for $-\frac{1}{2} < \nu < 0$.

Lemma 4.1. *We have*

$$\left(\kappa + \frac{\nu}{2} - \frac{1}{4}\right)\pi < j_{\nu\kappa} < \left(\kappa + \frac{\nu}{4} - \frac{1}{8}\right)\pi, \ \ \kappa \ge \frac{1}{2}, \tag{4.3}$$

for $-\frac{1}{2} < \nu < \frac{1}{2}$. The lower bound becomes exact for $\nu = -\frac{1}{2}$ and both bounds become exact for $\nu = \frac{1}{2}$. The inequalities are reversed for $\nu > \frac{1}{2}$.

Proof. The lower bound in the case $-\frac{1}{2} < \nu < \frac{1}{2}$ and upper bound in the case $\nu > \frac{1}{2}$ are well known and holds even for $\kappa > 0$. (See the work of A. Laforgia [12], also reported in [3]. More precise results are given in [7].) Now $j_{\nu\kappa}$ is a positive increasing function of κ. Further it is convex in κ for $-\frac{1}{2} < \nu < \frac{1}{2}$ and concave for $\nu > \frac{1}{2}$. Thus it is sufficient to prove the remaining bounds for $\kappa = \frac{1}{2}$, that is

$$y_{\nu 1} < \left(\frac{\nu}{4} + \frac{3}{8}\right)\pi, \ \ -\frac{1}{2} < \nu < \frac{1}{2}, \tag{4.4}$$

and

$$y_{\nu 1} > \left(\frac{\nu}{4} + \frac{3}{8}\right)\pi, \ \ \nu > \frac{1}{2}. \tag{4.5}$$

Now if we write

$$g(\nu) = y_{\nu 1} - \left(\frac{\nu}{4} + \frac{3}{8}\right)\pi,$$

we have $g'(\nu) = dy'_{\nu 1}/d\nu - \pi/4$ which, in view of results in [4] decreases to the positive number $1 - \pi/4$, as $\nu \to \infty$. Thus $g(\nu)$ is increasing, $g(-\frac{1}{2}) = -\frac{\pi}{4}$, $g(\frac{1}{2}) = 0$ and $g(\nu) \to \infty$ as $\nu \to \infty$. This gives the inequalities (4.4) and (4.5). □

Watson [15, pp. 490-491] has this result, but is not explicit about what would be, in our terms, the lower bound on κ.

5. Monotonic sequences related to Bessel functions

It is known [15, p. 446] that $p_\nu(x)$, as given by (4.2), is an increasing or decreasing function of x, $0 < x < \infty$, according to whether $0 \le \nu < \frac{1}{2}$ or $\frac{1}{2} < \nu < \infty$. Hence, we have the following:

Remark 5.1. *$\{j_{\nu k}^{1/2} Y_\nu(j_{\nu k})\}$ and $\{y_{\nu k}^{1/2} J_\nu(y_{\nu k})\}$ are alternating sequences of numbers whose absolute values increase to $\sqrt{2/\pi}$ for $0 \le \nu < \frac{1}{2}$ and decrease to $\sqrt{2/\pi}$ for $\nu > \frac{1}{2}$.*

In addition, we know [15, p.446] that $p_\nu(x)/x$ is a decreasing function of x on $(0,\infty)$ for each fixed $\nu \geq 0$. This shows that $|J_\nu(y_{\nu k})|$ and $|Y_\nu(j_{\nu k})|$ form decreasing sequences. More generally, if we have two cylinder functions of the same order, $\mathcal{C}_\nu(x,\alpha)$ and $\mathcal{C}_\nu(x,\beta)$, we have

$$\mathcal{C}_\nu(x,\beta) = \cos(\alpha-\beta)\mathcal{C}_\nu(x,\alpha) + \sin(\alpha-\beta)\mathcal{C}_\nu(x,\alpha-\pi/2). \tag{5.1}$$

If we use $\{x_k\}$ for the zeros of $\mathcal{C}_\nu(x,\alpha)$, we see that

$$\mathcal{C}_\nu(x_k,\beta) = \sin(\alpha-\beta)\mathcal{C}_\nu(x_k,\alpha-\pi/2). \tag{5.2}$$

In particular, we have, with $\alpha = \pi/2$,

$$\mathcal{C}_\nu(y_{\nu k},\beta) = \cos(\beta)J_\nu(y_{\nu k}) \tag{5.3}$$

and with $\alpha = 0$,

$$\mathcal{C}_\nu(j_{\nu k},\beta) = \sin(\beta)Y_\nu(j_{\nu k}). \tag{5.4}$$

Hence the monotonicity results for the special cases $\alpha = 0, \frac{\pi}{2}$ lead to similar results in the case of general α:

Remark 5.2. *For $\nu \geq 0$ and $0 \leq \alpha < \pi$, the sequences $|\mathcal{C}_\nu(j_{\nu k},\alpha)|$ and $|\mathcal{C}_\nu(y_{\nu k},\alpha)|$ decrease to 0 except in the trivial cases ($\alpha = 0$ for the first and $\alpha = \pi/2$ for the second) where they are identically 0.*

Now we consider a second equation

$$Y'' + \left[1 + \frac{\frac{1}{4}-\mu^2}{x^2}\right]Y = 0, \tag{5.5}$$

with solutions $Y_1(x) = \sqrt{\pi x/2}J_\mu(x)$ and $Y_2(x) = \sqrt{\pi x/2}Y_\mu(x)$, with

$$P(x) = p_\mu(x) = \frac{\pi}{2}x\left[J_\mu^2(x) + Y_\mu^2(x)\right]. \tag{5.6}$$

To compare p_μ and p_ν we use Nicholson's integral representation [15, p. 444, (1)]

$$p_\nu(x) = \frac{4}{\pi}x\int_0^\infty K_0(2x\sinh t)\cosh(2\nu t)\,dt. \tag{5.7}$$

This formula and related ones have been found useful in many investigations of monotonicity properties of Bessel function and their zeros; see [3, §1] and references.

In view of (5.7), we have $p_\mu(x) > p_\nu(x)$, when $\mu > \nu \geq 0$, so Theorem 3.1 gives:

Theorem 5.1. *Define*

$$f_\nu(x) = x^{1/2}\mathcal{C}_\nu(x,\alpha)/\sqrt{p_\nu(x)}. \tag{5.8}$$

Let $c_{\mu k}(\beta)$ be the kth positive zero of $\mathcal{C}_\mu(x,\beta)$, where $\mu > \nu \geq 0$, and suppose that for some positive integer k_0,

$$c_{\nu,k+m}(\alpha) < c_{\mu,k}(\beta) \leq c_{\nu,k+m+1}(\alpha+\pi/2), \quad k = k_0, k_0+1, \dots, \tag{5.9}$$

where m is the integer part of $(\alpha-\beta)/\pi + (\mu-\nu)/2$. Then the sequence $f_\nu(c_{\mu,k})$ is alternating and $|f_\nu(c_{\mu,k})|$ increases to its limiting value

$$L = \sqrt{\frac{2}{\pi}}\left|\sin\left(\frac{\mu-\nu}{2}\pi + \alpha - \beta\right)\right|, \tag{5.10}$$

as k increases from k_0 to ∞. If (5.9) is replaced by

$$c_{\nu,k+m+1}(\alpha+\pi/2) < c_{\mu,k}(\beta) \leq c_{\nu,k+m+1}(\alpha), \quad k = k_0, k_0+1, \dots, \tag{5.11}$$

then the sequence $f_\nu(c_{\mu,k})$ *is alternating and* $|f_\nu(c_{\mu,k})|$ *decreases to* L *as* k *increases from* k_0 *to* ∞*. The monotonicities are reversed if* $\nu > \mu \geq 0$.

The reason for the choice of m is that the asymptotic formula [15, p. 506]

$$c_{\nu k}(\alpha) \sim \left(k + \frac{\nu}{2} - \frac{1}{4}\right)\pi - \alpha,\ k \to \infty, \tag{5.12}$$

shows that in order for (5.9) to hold for *some* m, it is necessary that

$$\frac{\alpha-\beta}{\pi} + \frac{\mu-\nu}{2} - \frac{1}{2} < m < \frac{\alpha-\beta}{\pi} + \frac{\mu-\nu}{2},$$

and in order (5.11) to hold for *some* m, it is necessary that

$$\frac{\alpha-\beta}{\pi} + \frac{\mu-\nu}{2} - 1 < m < \frac{\alpha-\beta}{\pi} + \frac{\mu-\nu}{2} - \frac{1}{2}.$$

Here we look at some specific cases of Theorem 5.1, where k_0 can be chosen equal to 1, and hence we get monotonicity from the start.

Theorem 5.2. *Suppose that either*

$$0 \leq \nu < \mu \leq \frac{1}{2}, \tag{5.13}$$

or

$$\frac{1}{2} \leq \nu < \mu \leq \frac{3}{2}. \tag{5.14}$$

Then

$$\frac{(-1)^k j_{\mu k}^{1/2} J_\nu(j_{\mu k})}{\sqrt{p_\nu(j_{\mu k})}} \ \textit{increases to} \ \sqrt{\frac{2}{\pi}} \sin\left(\frac{(\mu-\nu)\pi}{2}\right) = L_1, \tag{5.15}$$

$$\frac{(-1)^{k+1} y_{\mu k}^{1/2} J_\nu(y_{\mu k})}{\sqrt{p_\nu(y_{\mu k})}} \ \textit{decreases to} \ \sqrt{\frac{2}{\pi}} \cos\left(\frac{(\mu-\nu)\pi}{2}\right) = L_2, \tag{5.16}$$

$$\frac{(-1)^{k+1} j_{\mu k}^{1/2} Y_\nu(j_{\mu k})}{\sqrt{p_\nu(j_{\mu k})}} \ \textit{decreases to} \ \sqrt{\frac{2}{\pi}} \cos\left(\frac{(\mu-\nu)\pi}{2}\right), \tag{5.17}$$

and

$$\frac{(-1)^{k+1} y_{\mu k}^{1/2} Y_\nu(y_{\mu k})}{\sqrt{p_\nu(y_{\mu k})}} \ \textit{increases to} \ \sqrt{\frac{2}{\pi}} \sin\left(\frac{(\mu-\nu)\pi}{2}\right), \tag{5.18}$$

as $k\,(= 1, 2, \dots)$ *increases to infinity. The monotonicities are reversed if either* $0 \leq \mu < \nu \leq \frac{1}{2}$ *or* $\frac{1}{2} \leq \mu < \nu \leq \frac{3}{2}$.

Corollary 5.3. *Under the hypotheses of Theorem 5.2,* $J_\nu^2(j_{\mu k})/Y_\nu^2(j_{\mu k})$ *and* $Y_\nu^2(y_{\mu k})/J_\nu^2(y_{\mu k})$ *increase to the limiting value* $\tan^2((\mu-\nu)\pi/2)$*, as* $k\ (= 1, 2, \dots)$ *increases.*

Proofs. The representation (5.7) gives the condition corresponding to (3.1), so to check the applicability of Theorem 3.1 we need to verify that

$$j_{\nu k} < j_{\mu k} \leq y_{\nu, k+1},\ \ k = 1, 2, \dots \tag{5.19}$$

the condition corresponding to (3.2). The first inequality here follows from the increasing nature of $j_{\nu k}$ as a function of ν, $\nu > -1$. The second inequality can be expresses as $j_{\mu\kappa} \leq j_{\nu,\kappa+\frac{1}{2}}$ which follows from Lemma 4.1, provided $\mu \leq 2\nu + \frac{3}{2}$

in case (5.13) holds, or $2\mu \leq \nu + \frac{5}{2}$ in case (5.14) holds. These are both easy consequences of the hypotheses on μ and ν. To get the correct signs, we also need to show that $J_\nu(j_{\mu 1}) < 0$. This follows from the consequence $j_{\nu 1} < j_{\mu 1} < y_{\nu 2}$ of (5.19). Hence (5.15) holds. The result (5.16) is proved similarly, the inequalities (5.19) being replaced by

$$y_{\nu,k+1} < y_{\mu,k+1} \leq j_{\nu,k+1}, \quad k = 1, 2, \ldots. \tag{5.20}$$

Similar remarks apply to (5.17) and (5.18). The limits of the sequences follow from the asymptotic behaviour of the Bessel functions and their zeros [15, Chapter 7].

To see that the the monotonicities are reversed in case $\frac{1}{2} \geq \nu > \mu \geq 0$, we consider that in that case we have

$$y_{\nu k} < j_{\mu k} < j_{\nu k} \tag{5.21}$$

and

$$j_{\nu,k-1} < y_{\mu k} < y_{\nu k}. \tag{5.22}$$

Corollary 5.3 follows from the Theorem on using formula (4.2) for $p_\nu(x)$.

5.1. Cylinder functions of order between $-\frac{1}{2}$ and $\frac{1}{2}$.

Corollary 5.4. *If* $0 \leq \nu < \mu \leq \frac{1}{2}$ *then* $(-1)^k \sqrt{j_{\mu k}} J_\nu(j_{\mu k})$ *and* $(-1)^{k+1} \sqrt{y_{\mu k}} Y_\nu(y_{\mu k})$ *increase to the limiting value* L_1*, given by (5.15), as* k $(= 1, 2, \ldots)$ *increases.*

Corollary 5.4 follows from the first part of Theorem (5.2) on noting that, for $0 \leq \nu < \frac{1}{2}$, $p_\nu(x)$ increases [15, p. 446] to 1 as x increases.

Corollary 5.5. *For* $0 \leq \nu < \frac{1}{2}$, $(-1)^k \sqrt{k} J_\nu(k\pi)$ *and* $(-1)^{k+1} \sqrt{k - \frac{1}{2}} Y_\nu((k - \frac{1}{2})\pi)$ *increase to the limiting value* $(\sqrt{2}/\pi) \sin((1 - 2\nu)\pi/4)$ *as* k $(= 1, 2, \ldots)$ *increases. In particular, with* $\nu = 0$*, this gives*

$$0 < (-1)^k J_0(k\pi) < \frac{1}{\pi\sqrt{k}}, \quad k = 1, 2, \ldots. \tag{5.23}$$

and

$$0 < (-1)^{k+1} Y_0((k - \frac{1}{2})\pi) < \frac{1}{\pi\sqrt{k - \frac{1}{2}}}, \quad k = 1, 2, \ldots. \tag{5.24}$$

Corollary 5.5 follows from Corollary 5.4 on taking $\mu = \frac{1}{2}$.

The consequence $|J_0(k\pi)| < 1/(\pi\sqrt{k})$ of the left-hand inequality in (5.23) was proved (with $\leq$) for m even in [1, Lemma 2].

5.2. Cylinder Functions of order between $\frac{1}{2}$ and $\frac{3}{2}$.

Corollary 5.6. *Suppose that* $\frac{1}{2} \leq \nu < \mu \leq \frac{3}{2}$*. Then* $(-1)^{k+1} \sqrt{y_{\mu k}} J_\nu(y_{\mu k})$ *and* $(-1)^{k+1} \sqrt{j_{\mu k}} Y_\nu(j_{\mu k})$ *decrease to the limiting value* L_2*, given by (5.16), as* k $(= 1, 2, \ldots)$ *increases.*

This follows from (5.16), (5.17), since $p_\nu(x)$ is decreasing.

Corollary 5.7. *Suppose that* $\frac{1}{2} < \nu \leq \frac{3}{2}$*. Then* $(-1)^{k+1} \sqrt{k} J_\nu(k\pi)$ *and* $(-1)^k \sqrt{k - \frac{1}{2}} Y_\nu((k - \frac{1}{2})\pi)$ *increase to the limiting value* $(\sqrt{2}/\pi) \cos((2\nu - 1)\pi/4)$ *as* k $(= 1, 2, \ldots)$ *increases.*

Clearly the condition corresponding to (3.1) holds so to check the applicability of Theorem 3.1 we need to verify that

$$y_{\nu k} < j_{\mu k} < j_{\nu k}, \quad k = 1, 2, \ldots. \tag{5.25}$$

the condition corresponding to (3.2). The second inequality here follows from the increasing nature of $j_{\nu k}$ as a function of ν, $\nu > -1$. To show the left-hand inequality in (5.25), it suffices, in view of Lemma 4.1, to verify that

$$k\pi - \frac{3}{4}\pi + \frac{1}{2}\nu\pi \le k\pi - \frac{1}{8}\pi + \frac{1}{4}\mu\pi,$$

or that $\mu \ge 2\nu - \frac{5}{2}$, an easy consequence of the hypotheses on μ and ν. The result concerning Y_ν is proved similarly, the inequalities (5.25) being replaced by

$$j_{\nu,k} < y_{\mu,k+1} \le y_{\nu,k+1}, \quad k = 1, 2, \ldots. \tag{5.26}$$

The right-hand inequality is obvious and the left-hand one follows as before.

Now we expand a little on the situation where the "monotonicities are reversed". We suppose that either

$$0 \le \mu < \nu \le \frac{1}{2}, \tag{5.27}$$

or that,

$$\frac{1}{2} \le \mu < \nu \le \frac{3}{2}. \tag{5.28}$$

Then the reversed form of 5.15) refers to a sequence of negative numbers *decreasing* to a negative limit and it can be expressed as

$$\frac{(-1)^{k+1} j_{\mu k}^{1/2} J_\nu(j_{\mu k})}{\sqrt{p_\nu(j_{\mu k})}} \ \textit{increases to} \ \sqrt{\frac{2}{\pi}} \sin\left(\frac{(\nu-\mu)\pi}{2}\right) \tag{5.29}$$

In the particular case where $\nu = 1$ Corollary 5.7 gives:

Corollary 5.8. *The positive quantities $(-1)^{k+1} k^{1/2} J_1(k\pi)$ increase to $1/\pi$ as k ($= 1, 2, \ldots$) increases. Hence*

$$0 < (-1)^{k+1} J_1(k\pi) < \frac{1}{\pi\sqrt{k}}, \quad k = 1, 2, \ldots, \tag{5.30}$$

in analogy to (1.2).

5.3. **Values of cylinder functions at multiples of π.** The above results deal with situations where μ and ν differ by at most 1. Numerical evidence indicates that we cannot expect monotonicity, though we might expect ultimate monotonicity, when μ and ν are further apart. We confine ourselves here to examining further the situations arising in Corollaries 5.5 and 5.7, which deal with the values of a cylinder function at the points $k\pi$, where k runs through a sequence of integers. For such sequences, we will be interested in estimating the point at which they become monotonic.

Theorem 5.9. *Let*

$$n - \frac{1}{2} < \nu < n + \frac{1}{2},$$

for some positive integer n. In case $n = 2s + 1$ is odd, let

$$y_{\nu k} < (s + k)\pi < j_{\nu k}, \quad k \ge k_0, \tag{5.31}$$

for some positive integer k_0. *Let*

$$s_k = k^{1/2} J_\nu(k\pi)/\sqrt{p_\nu(k\pi)},\ \ t_k = k^{1/2} Y_\nu(k\pi)/\sqrt{p_\nu(k\pi)},\ k \geq k_0 + s,$$

Then $\{s_k\}$ *and* $\{t_k\}$ *are alternating sequences,* $|s_k|$ *increases to* $(\sqrt{2}/\pi)|\cos((2\nu+1)\pi/4)|$ *and* $|t_k|$ *decreases to* $(\sqrt{2}/\pi)|\sin((2\nu+1)\pi/4)|$ *as* $k(\geq k_0+s)$ *increases.*

In case $n = 2s$ *is even and (5.31) is replaced by*

$$j_{\nu k} < (s+k)\pi < y_{\nu,k+1},\ \ k \geq k_0, \tag{5.32}$$

the monotonicities of $|s_k|$ *and* $|t_k|$ *are reversed.*

Proof. This is a straightforward application of Theorem 3.1 to the equations (4.1) and (5.5) with $\mu = \frac{1}{2}$. The inequality (3.1) is reversed and (3.2) is replaced by (3.3) for the J case. In the Y case, we use the same Theorem with $y(x,\alpha) = x^{1/2}Y_\nu(x),\ \ y(x, \alpha - \pi/2) = -x^{1/2}J_\nu(x)$.

Corollary 5.10. *Let* $\nu = 2m+1$ *be an odd positive integer and suppose that*

$$j_{\nu,k_1} > \left(\frac{\nu}{2} - \frac{1}{2} + k_1\right)\pi, \tag{5.33}$$

for some positive integer k_1. *Then*

$$\frac{(-1)^{k-m+1}k^{1/2}J_\nu(k\pi)}{\sqrt{p_\nu(k\pi)}} \text{ increases to } 1/\pi,$$

$$\frac{(-1)^{k-m+1}k^{1/2}Y_\nu(k\pi)}{\sqrt{p_\nu(k\pi)}} \text{ decreases to } 1/\pi,$$

and hence also

$$(-1)^{k-m+1}k^{1/2}Y_\nu(k\pi) \text{ decreases to } 1/\pi,$$

as $k\ (\geq k_1 + m)$ *increases.*

Proof. We have to verify that the conditions (5.31) are satisfied. The left-hand inequality is the consequence $y_{\nu k} < k + \nu/2 - 3/4$ of Lemma 4.1. In view of the concavity of $j_{\nu\kappa}$ as a function of κ, the inequality $j_{\nu\kappa} > (m+k)\pi$ for $\kappa = k_1$ (assumption (5.33), will show it to be true for all $\kappa > k_1$. □

For positive even integer values of ν, we get:

Corollary 5.11. *Let* $\nu = 2m$ *be an even positive integer and suppose that*

$$y_{\nu,k_1+1} > \left(\frac{\nu}{2} + k_1\right)\pi, \tag{5.34}$$

for some positive integer k_1. *Then*

$$\frac{(-1)^{k-m+1}k^{1/2}Y_\nu(k\pi)}{\sqrt{p_\nu(k\pi)}} \text{ increases to } 1/\pi,$$

$$\frac{(-1)^{k-m}k^{1/2}J_\nu(k\pi)}{\sqrt{p_\nu(k\pi)}} \text{ decreases to } 1/\pi,$$

and hence also

$$(-1)^{k-m}k^{1/2}J_\nu(k\pi) \text{ decreases to } 1/\pi,$$

as $k\ \ (\geq k_1 + m)$ *increases.*

The proof is similar to that of Corollary 5.10.

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